

Metric Symplectic Lie Algebras*

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Every metric symplectic Lie algebra has the structure of a quadratic extension. We give a standard model and describe the equivalence classes on the level of corresponding quadratic cohomology sets. Finally, we give a scheme to classify the isomorphism classes of metric symplectic Lie algebras and give a complete list of all these Lie algebras in special cases.

1 Introduction

In this paper we concentrate on metric symplectic Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$. These are symplectic Lie algebras (\mathfrak{g}, ω) which are also metric Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$. Of course, an isomorphism of metric symplectic Lie algebras is an isomorphism of the corresponding symplectic Lie algebras which is also an isomorphism of the corresponding metric Lie algebras. We call a metric symplectic Lie algebra decomposable if it is isomorphic to the direct sum of two non-trivial metric symplectic Lie algebras.

Here a symplectic Lie algebra (\mathfrak{g}, ω) is a Lie algebra \mathfrak{g} admitting a closed non-degenerate 2-form ω on \mathfrak{g} , which we call symplectic form. Two symplectic Lie algebras $(\mathfrak{g}_i, \omega_i)$, $i = 1, 2$ are isomorphic if there is an isomorphism $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of Lie algebras which preserves the symplectic forms in the sense $\varphi^* \omega_2 = \omega_1$. Symplectic Lie algebras are in one-to-one correspondence with simply connected Lie groups with leftinvariant symplectic forms. Symplectic Lie algebras are also called quasi-Frobenius Lie algebras, since Frobenius Lie Algebras, i. e. Lie algebras admitting an non-degenerate exact 2-form, are examples of symplectic Lie algebras. Moreover, symplectic Lie algebras are examples of Vinberg algebras, since they naturally carry an affine structure.

A metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is a Lie algebra \mathfrak{g} with an non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} , which is ad-invariant, i. e.

$$\langle [X, Y], Z \rangle = \langle X, [Y, Z] \rangle$$

for all $X, Y, Z \in \mathfrak{g}$. Although, $\langle \cdot, \cdot \rangle$ is not necessary positive definite, we will call it inner product. In the literature these Lie algebras are also called quadratic Lie algebras. Two

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metric Lie algebras are isomorphic if there is an isomorphism of the corresponding Lie algebras, which is also an isometry of the inner products.

Metric Lie algebras are in one-to-one correspondence with simply connected Lie groups with biinvariant metrics. They are also used for describing symmetric spaces, since the Lie algebra of the transvection group of a symmetric space admits such a structure [5, Proposition 1.6].

There are several classifications of metric or symplectic Lie algebras in low dimension ([20], [9], [19], [6]). Especially, the nilpotent case ([4], [12], [15], see also [10]) is important for our problem, since every metric symplectic Lie algebra is nilpotent.

If the aim is to give informations for arbitrary dimension, then usually a reduction scheme is used. In this context we mention double extensions ([16],[17]), T^* -extensions [3], symplectic double extensions ([18], [8], [7]) and oxidation [2]. The main idea is, that for every isotropic ideal \mathfrak{j} of a metric or symplectic Lie algebra \mathfrak{g} the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{j}^\perp/\mathfrak{j}$ inherits an inner product or symplectic form respectively from \mathfrak{g} . Conversely, they give a construction scheme taking a metric or symplectic Lie algebra $\bar{\mathfrak{g}}$ and some additional structure and construct a higher dimensional metric respectively symplectic Lie algebra, which can be reduced to $\bar{\mathfrak{g}}$ again. Since the choice of the isotropic ideal is in general not canonical, it is hard to give a general statement on the isomorphy of this Lie algebras with the help of these schemes. For instance, it is possible that extensions of two non-isomorphic low dimensional metric or symplectic Lie algebras are isomorphic. Moreover, the presentation of a metric or symplectic Lie algebra as such an extension is not unique, since it depends on the choosen ideal.

The aim of this paper is to take this choices canonically for metric symplectic Lie algebras such that there is a certain standard model and the possibility to analyse the isomorphy systematically with this standard model.

Until now, there is just a few literature about metric symplectic Lie algebras (for example [1]). These Lie algebras are in one-to-one correspondence to nilpotent metric Lie algebras with bijective skewsymmetric derivations. On every Lie group with Lie algebra \mathfrak{g} there is a flat and torsion-free connection ∇^ω induced by ω and an left-invariant pseudo-Riemannian metric whose Levi-Civita connection equals ∇^ω [18].

A classification scheme for metric Lie algebras from Kath and Olbrich denoted as quadratic extension is very useful for our aim. This scheme was first introduced in [13], where metric Lie algebras were discussed comprehensively. Using the main idea of [13] they introduced a classification scheme in [14], which can be used for metric Lie algebras and metric Lie algebras with additional structure. Especially, metric Lie algebras with semisimple skewsymmetric derivations were treated in that paper in the notion of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras. Thus, this is useful for our problem. Our aim is to expand the classification scheme in [14] for metric Lie algebras with semisimple skewsymmetric derivations in such a way that it classifies all metric symplectic Lie algebras.

In the following, we describe the main idea of the present paper. For every metric

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Lie algebra \mathfrak{g} with bijective skewsymmetric derivation D there is an D -invariant isotropic ideal \mathfrak{i} of \mathfrak{g} in a canonical way such that $\mathfrak{i}^\perp/\mathfrak{i}$ is abelian. This ideal is given by

$$\mathfrak{i} = \mathfrak{i}(\mathfrak{g}) := \sum_{k=0}^{n-1} \mathfrak{g}^{k+1} \cap \mathfrak{g}^{k+1\perp} \quad (1)$$

(compare [12]). Here $\mathfrak{g}^1 := \mathfrak{g}, \dots, \mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$ denotes the decreasing central series of the nilpotent Lie algebra \mathfrak{g} and n the smallest positive integer such that $\mathfrak{g}^n = \{0\}$. For the definition of this ideal in the case of non-nilpotent metric Lie algebras see [13] or [14]. We set $\mathfrak{a} = \mathfrak{i}^\perp/\mathfrak{i}$ and $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}^\perp$ and obtain that \mathfrak{a} is an abelian metric symplectic Lie algebra and inherits the structure of a trivial \mathfrak{l} -module. Since \mathfrak{i} is isomorphic to \mathfrak{l}^* , we can write \mathfrak{g} as two abelian extensions of Lie algebras with bijective derivations

$$0 \rightarrow \mathfrak{l}^* \rightarrow \mathfrak{g} \rightarrow \mathfrak{h} \rightarrow 0, \quad 0 \rightarrow \mathfrak{a} \rightarrow \mathfrak{h} \rightarrow \mathfrak{l} \rightarrow 0 \quad (2)$$

in a canonical way. Here $\mathfrak{h} = \mathfrak{g}/\mathfrak{i}$. Conversely, two abelian extensions given as in (2) define a metric symplectic Lie algebra for every Lie algebra \mathfrak{l} with bijective derivation and \mathfrak{l} -module \mathfrak{a} , unless the abelian extensions satisfy certain compatibility conditions. Then the image of \mathfrak{l}^* in \mathfrak{g} is usually not equal to the canonical isotropic ideal.

We will introduce this construction scheme under the notion of quadratic extensions. In general, not every quadratic extension of \mathfrak{l} by \mathfrak{a} is given by the canonical isotropic ideal $\mathfrak{i}(\mathfrak{g})$. Thus we also introduce balanced quadratic extensions. We call a quadratic extension balanced, if the image of \mathfrak{l}^* in \mathfrak{g} equals the canonical isotropic ideal. Every metric symplectic Lie algebra has the structure of a balanced quadratic extension in a canonical way. There is a natural equivalence relation on the set of quadratic extensions of \mathfrak{l} by \mathfrak{a} . Moreover, we define a non-linear cohomology $H_{Q+}^p(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of the Lie algebra \mathfrak{l} with coefficients in \mathfrak{a} , which includes the cohomology $H_Q^p(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ considered in [14]. Then we prove that the equivalence classes of quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are in bijection to the second cohomology set $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. For the proof we cite necessary results for metric Lie algebras with semisimple skewsymmetric derivations and fit this to the case of not necessary semisimple derivations. Moreover, we give the standard model $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ of a metric Lie algebra with skewsymmetric derivation, which defines a quadratic extension of \mathfrak{l} by \mathfrak{a} for every $[\alpha, \gamma, \delta, \epsilon] \in H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

The equivalence classes of balanced quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ are described by $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b \subset H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ on the level of the cohomology classes. We also call these cocycles balanced. Since balanced is a property of a quadratic extension which does only depend on the structure of the corresponding metric Lie algebra, we can use [13] for describing balanced cocycles, or more precisely [12] for nilpotent, metric Lie algebras. We also have the notion of indecomposable balanced cohomology classes, contained in $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 \subset H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$. The automorphism group $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ of the pair $((\mathfrak{l}, D_{\mathfrak{l}}), \mathfrak{a})$ acts on $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$. Altogether, the isomorphism classes of metric, symplectic Lie algebras

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are in one-to-one correspondence to

$$\coprod_{(l, D_l) \in \mathcal{L}} \coprod_{\alpha \in \mathcal{A}_{l, D_l}} H_{Q+}^2(l, D_l, \alpha)_0 / G(l, D_l, \alpha),$$

where \mathcal{L} is a system of representatives of the isomorphism classes of nilpotent Lie algebras with bijective derivations and \mathcal{A}_{l, D_l} a system of representatives of the isomorphism classes of abelian metric Lie algebras with bijective skewsymmetric derivations (considered as trivial l -modules). Using this classification scheme we obtain the following results:

- There is only one non-abelian metric Lie algebra (up to isomorphism) of dimension less than eight, which admits symplectic forms.
- There are no metric symplectic Lie algebras whose inner product has an index of one or two, except for abelian ones.
- We calculate all isomorphism classes of metric symplectic Lie algebras of dimension less than ten and
- we calculate every non-abelian metric symplectic Lie algebra with an index of three up to isomorphism.

The paper is organized as follows. We introduce the notion of (balanced) quadratic extensions in section 3 and show that every metric symplectic Lie algebra has the structure of a (balanced) quadratic extension in a canonical way. Furthermore, we give an equivalence relation on the set of quadratic extensions. Then, we define the quadratic cohomology $H_{Q+}^p(l, D_l, \alpha)$ in section 4. We also define the isomorphism of pairs and give their action on the quadratic cohomology. In section 5 we define the standard model $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ and show necessary and sufficient conditions on the level of the corresponding cocycles, when $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ has the structure of a (balanced) quadratic extension. We show that every quadratic extension is equivalent to a suitable standard model (section 6) and describe the equivalence of standard models (section 7). In the end of section 7 we obtain a bijection between $H_{Q+}^2(l, D_l, \alpha)$ and the equivalence classes of quadratic extensions of l by α . In section 8 we describe the isomorphy of standard models on the level of the corresponding cohomology classes and obtain, finally, the classification scheme for metric symplectic Lie algebras. As an application, we calculate in section 9 all non-abelian metric symplectic Lie algebras (up to isomorphisms) whose index of the inner product is less than four. Moreover, we give a system of representatives of all non-abelian metric symplectic Lie algebras of dimension less than ten.

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Definition 1. A metric, symplectic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$ is a Lie algebra \mathfrak{g} with a non-degenerate (not necessary positive definite) symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ and

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a non-degenerate skewsymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ satisfying

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \quad \text{and}$$

$$d\omega(X, Y, Z) = -\omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Usually, we will call $\langle \cdot, \cdot \rangle$ inner product and ω symplectic form.

Definition 2. An isomorphism φ between metric symplectic Lie algebras $(\mathfrak{g}_1, \langle \cdot, \cdot \rangle_1, \omega_1)$ and $(\mathfrak{g}_2, \langle \cdot, \cdot \rangle_2, \omega_2)$ is an isomorphism $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of the corresponding Lie algebras, which is an isometry, i. e. $\varphi^* \langle \cdot, \cdot \rangle_2 = \langle \cdot, \cdot \rangle_1$, and satisfies $\varphi^* \omega_2 = \omega_1$.

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In this section we introduce the necessary notion of quadratic extensions for the new classification scheme for metric symplectic Lie algebras. Moreover, we show that every metric symplectic Lie algebra has the structure of a quadratic extension in a canonical way.

Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle, \omega)$ be a metric symplectic Lie algebra. Then

$$\omega = \langle \cdot, D \cdot \rangle \tag{3}$$

defines a bijective skewsymmetric map D on \mathfrak{g} , since $\langle \cdot, \cdot \rangle$ is non-degenerate. Here D is called skewsymmetric, if $\langle DX, Y \rangle = -\langle X, DY \rangle$ for all $X, Y \in \mathfrak{g}$. Moreover, D is a derivation on \mathfrak{g} , since ω is closed. Now, the existence of a bijective derivation on \mathfrak{g} implies, that \mathfrak{g} is nilpotent [11, Theorem 3]. Conversely, every bijective skewsymmetric derivation of a nilpotent metric Lie-Algebra \mathfrak{g} defines a symplectic form ω by equation (3). Thus we obtain the following Lemma.

Lemma 3. *Metric symplectic Lie algebras are in one-to-one correspondence with nilpotent metric Lie-Algebras with skewsymmetric bijective derivations.*

From now on, we concentrate on metric Lie algebras $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ with skewsymmetric derivation D and write abbreviatory $(\mathfrak{g}, \langle \cdot, \cdot \rangle, D)$, (\mathfrak{g}, D) or simply \mathfrak{g} unless it is clear from the context that this is a metric Lie algebra with a skewsymmetric derivation.

Definition 4. An isomorphism (homomorphism) φ between Lie algebras with derivations (\mathfrak{g}_1, D_1) and (\mathfrak{g}_2, D_2) is an isomorphism (homomorphism) $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of the corresponding Lie algebras, which satisfies $D_2 \varphi = \varphi D_1$. An isomorphism between metric Lie algebras with derivations is an isomorphism of Lie algebras with derivations, which is in addition an isometry.

Lemma 5. *Let (\mathfrak{g}, D) be a metric Lie algebra with derivation. Then the semisimple part D_s of the Jordan decomposition of the skewsymmetric derivation D is also a skewsymmetric derivation of \mathfrak{g} .*

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Proof. Consider the complexification of D and $\langle \cdot, \cdot \rangle$, if D has nonreal eigenvalues. Let v_1 and v_2 denote two generalized eigenvectors of D corresponding to the eigenvalues λ_1 and λ_2 . For a sufficiently large $k \in \mathbb{R}$ we have that

$$(D - (\lambda_1 + \lambda_2) \text{id})^k [v_1, v_2] = \sum_{i=0}^k \binom{k}{i} [(D - \lambda_1 \text{id})^{k-i} v_1, (D - \lambda_2 \text{id})^i v_2]$$

vanishes. Thus $[v_1, v_2]$ is an vector of the generalized eigenspace for $\lambda_1 + \lambda_2$ and we obtain $D_s[v_1, v_2] = (\lambda_1 + \lambda_2)[v_1, v_2]$. Thus, D_s is a derivation, since

$$D_s[v_1, v_2] = (\lambda_1 + \lambda_2)[v_1, v_2] = [\lambda_1 v_1, v_2] + [v_1, \lambda_2 v_2] = [D_s v_1, v_2] + [v_1, D_s v_2].$$

Since D is skewsymmetric, the generalized eigenspaces for λ_1 and λ_2 , $\lambda_1 \neq -\lambda_2$ are orthogonal to each other. This subspaces are invariant under D_s . Thus $\langle D_s v_1, v_2 \rangle + \langle v_1, D_s v_2 \rangle = 0$. For $\lambda_1 = -\lambda_2$ we obtain

$$\langle D_s v_1, v_2 \rangle + \langle v_1, D_s v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle - \lambda_1 \langle v_1, v_2 \rangle = 0.$$

Hence D_s is skewsymmetric. □

Definition 6. Let \mathfrak{l} be a Lie algebra. The triple $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ is called orthogonal \mathfrak{l} -module, if $(\mathfrak{a}, \langle \cdot, \cdot \rangle)$ is an abelian metric Lie algebra and $\rho : \mathfrak{l} \rightarrow \text{Hom}(\mathfrak{a})$ a representation of the Lie algebra \mathfrak{l} on \mathfrak{a} , which is skewsymmetric, i. e. $\langle \rho(L)A_1, A_2 \rangle = -\langle A_1, \rho(L)A_2 \rangle$ for all $L \in \mathfrak{l}$ and $A_1, A_2 \in \mathfrak{a}$.

Now, let $D_{\mathfrak{l}}$ be a derivation on \mathfrak{l} . We call $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module, if $D_{\mathfrak{a}}$ is a skewsymmetric map on \mathfrak{a} and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}})$ an orthogonal \mathfrak{l} -module satisfying $\rho(D_{\mathfrak{l}}L) = [D_{\mathfrak{a}}, \rho(L)]$ for all $L \in \mathfrak{l}$. We will also write $(\mathfrak{a}, D_{\mathfrak{a}})$ abbreviately for an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module.

Moreover, if ρ is the trivial representation of \mathfrak{l} on \mathfrak{a} , then we will call $(\mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module.

Lemma 7. Let \mathfrak{l} be a Lie algebra, $D_{\mathfrak{l}}$ a derivation on \mathfrak{l} and $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(\rho, \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{a}}, D_{\text{as}})$ is an orthogonal $(\mathfrak{l}, D_{\text{ls}})$ -module, where D_{ls} and D_{as} denote the semisimple parts of the Jordan decomposition of $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$.

Proof. Because of Lemma 5 it remains to show that

$$\rho(D_{\text{ls}}L) = [D_{\text{as}}, \rho(L)]$$

holds for all $L \in \mathfrak{l}$. Consider the complexification of $D_{\mathfrak{a}}$, $D_{\mathfrak{l}}$ and ρ if $D_{\mathfrak{a}}$ or $D_{\mathfrak{l}}$ have nonreal eigenvectors. Let $L \in \mathfrak{l}$ be a vector in the generalized eigenspace of $D_{\mathfrak{l}}$ for λ and $A \in \mathfrak{a}$ in

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the generalized eigenspace of D_α for μ . Similary to the proof of Lemma 5 we obtain

$$(D_\alpha - (\lambda + \mu) \text{id})^k \rho(L)A = \sum_{i=0}^k \binom{k}{i} \rho((D_l - \lambda \text{id})^{k-i} (D_\alpha - \mu \text{id})^i A).$$

Thus $(D_\alpha - (\lambda + \mu) \text{id})^k \rho(L)A$ vanishes for sufficiently large k . Hence $\rho(L)A$ is a vector in the generalized eigenspace for $\lambda + \mu$. Finally $D_{\alpha s} \rho(L)A = (\lambda + \mu) \rho(L)A$ and

$$D_{\alpha s} \rho(L)A - \rho(L)D_{\alpha s}A = \lambda \rho(L)A = \rho(D_{l s}L)A.$$

□

Definition 8. Let l be a Lie algebra, D_l a derivation of l and $(\rho, \alpha, \langle \cdot, \cdot \rangle, D_\alpha)$ an orthogonal (l, D_l) -module. A quadratic extension of (l, D_l) by (α, D_α) is a quadrupel (g, i, \bar{D}, p) , where

- $g = (g, \langle \cdot, \cdot \rangle, D)$ is a metric Lie algebra with skewsymmetric derivation,
- $i \subset g$ is an isotropic D -invariant ideal of g ,
- $i : (\alpha, D_\alpha) \rightarrow (g/i, \bar{D})$ and $p : (g/i, \bar{D}) \rightarrow (l, D_l)$ are homomorphisms of Lie algebras with derivations such that

$$0 \rightarrow \alpha \xrightarrow{i} g/i \xrightarrow{p} l \rightarrow 0$$

is a short exact sequence of Lie algebras. Here \bar{D} denotes the skewsymmetric derivation of g/i induced by D . Moreover,

$$i(\rho(L)A) = [\tilde{L}, i(A)] \in i(\alpha) \quad (4)$$

holds for all $\tilde{L} \in g/i$ satisfying $p(\tilde{L}) = L$. Furthermore, $\text{im}(i) = i^\perp/i$ and i is an isometry onto i^\perp/i .

Lemma 9. Let $(g, D; i, \bar{D}, p)$ be a quadratic extension of (l, D_l) by (α, D_α) . Then $(g, D_s; i, \bar{D}, p)$ is also a quadratic extension of $(l, D_{l s})$ by $(\alpha, D_{\alpha s})$. Here D_s , $D_{l s}$ and $D_{\alpha s}$ denote the semisimple parts of the Jordan decomposition of D , D_l and D_α .

Proof. It is well known that every D -invariant subspace is also invariant under the semisimple part D_s of the Jordan decomposition of D . So it is not hard to see that $i : (\alpha, D_\alpha) \rightarrow (g/i, \bar{D})$ and $p : (g/i, \bar{D}) \rightarrow (l, D_l)$ are homomorphisms of Lie algebras with corresponding semisimple derivations. □

Let i be an isotropic D -invariant ideal of a metric Lie algebra g with bijective skewsymmetric derivation D such that i^\perp/i is abelian. Then the short exact sequence

$$0 \rightarrow i^\perp/i \xrightarrow{i} g/i \xrightarrow{p} g/i^\perp \rightarrow 0 \quad (5)$$

defines a quadratic extension of g/i^\perp by i^\perp/i with corresponding induced derivations.

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Remark 10. *This was already proved for metric Lie algebras without additional structure in [13, Page 94]. In [14], this statement was generalized for metric Lie algebras with additional structure, the so called (\mathfrak{h}, K) -equivariant metric Lie algebras. This also includes metric Lie algebras with semisimple skewsymmetric derivations in the notation of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras. But it is not necessary that the derivations are semisimple to define a quadratic extension using the short exact sequence (5).*

We already know that nilpotent metric Lie algebras with bijective skewsymmetric derivations are the main structure to determine the isomorphism classes of metric symplectic Lie algebras. So, let \mathfrak{g} denote a nilpotent metric Lie algebra and D a skewsymmetric, bijective derivation of \mathfrak{g} for the rest of this section.

We are interested in quadratic extensions whose ideal is the canonical isotropic ideal

$$i(\mathfrak{g}) := \sum_{k=0}^{n-1} \mathfrak{g}^{k+1} \cap \mathfrak{g}^{k+1\perp}. \quad (6)$$

Here $\mathfrak{g}^1 := \mathfrak{g}, \dots, \mathfrak{g}^k := [\mathfrak{g}, \mathfrak{g}^{k-1}]$ denotes the lower central series of \mathfrak{g} and n the smallest positive integer such that $\mathfrak{g}^n = \{0\}$. This is the definition of the canonical isotropic ideal for nilpotent \mathfrak{g} as it was already used in [12]. The definition of $i(\mathfrak{g})$ for a not necessary nilpotent \mathfrak{g} is given in [13, Definition 3.3].

This ideal is isotropic for every nilpotent metric Lie algebra \mathfrak{g} , it is D -invariant for every derivation D of \mathfrak{g} and moreover i^\perp/i is abelian [13, Lemma 3.4 (d)] [14, Proposition 2.7]. Thus we will call this ideal $i(\mathfrak{g})$ the canonical isotropic ideal. Furthermore, it is not hard to prove that every isometry $F : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ of nilpotent metric Lie algebras satisfies $F(i(\mathfrak{g}_1)) = i(\mathfrak{g}_2)$.

Lemma 11. ([12]) *Let \mathfrak{g} be a nilpotent metric Lie algebra and $i(\mathfrak{g})$ the canonical isotropic ideal. If $(\mathfrak{g}, i(\mathfrak{g}), i, p)$ is a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$, then $(\mathfrak{a}, D_{\mathfrak{a}})$ is a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module, i. e. ρ is trivial.*

Definition 12. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle, D)$ be a nilpotent metric Lie algebra with bijective skewsymmetric derivation D . Moreover, let \mathfrak{l} be a nilpotent Lie algebra, $D_{\mathfrak{l}}$ a bijective derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module, where $D_{\mathfrak{a}}$ is bijective. A quadratic extension (\mathfrak{g}, i, p) of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$ is called balanced, if i is the canonical isotropic ideal $i = i(\mathfrak{g})$.

Theorem 1. *Every nilpotent metric Lie algebra \mathfrak{g} with skewsymmetric bijective derivation D has the structure of a balanced quadratic extension. I. e., there is a nilpotent Lie algebra \mathfrak{l} with bijective derivation $D_{\mathfrak{l}}$, an abelian metric Lie algebra \mathfrak{a} with bijective skewsymmetric derivation $D_{\mathfrak{a}}$ considered as a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module and homomorphisms i and p of Lie algebras such that $(\mathfrak{g}, i(\mathfrak{g}), i, p)$ is a balanced quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$.*

Proof. We choose $i = i(\mathfrak{g})$. This ideal is isotropic, D -invariant and $i(\mathfrak{g})^\perp/i(\mathfrak{g})$ is abelian. Thus the sequence (5) defines a quadratic extension with corresponding derivations, which is balanced by definition. \square

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It is also possible that (\mathfrak{g}, i, i, p) is a balanced quadratic extension of the trivial Lie algebra $\{0\}$ by α . This means that $i(\mathfrak{g}) = \{0\}$, which is equivalent to \mathfrak{g} is abelian. Thus a non-abelian \mathfrak{g} has the structure of a non-trivial balanced quadratic extension.

There is a natural equivalence relation on the set of quadratic extensions.

Definition 13. Two quadratic extensions $(\mathfrak{g}_j, D_j; i_j, i_j, p_j)$, $j = 1, 2$ of (\mathfrak{l}, D_1) by (α, D_α) are equivalent, if there is an isomorphism $F : (\mathfrak{g}_1, D_1) \rightarrow (\mathfrak{g}_2, D_2)$ of the metric Lie algebras with derivations such that $F(i_1) = i_2$ and the induced isomorphism $\bar{F} : \mathfrak{g}_1/i_1 \rightarrow \mathfrak{g}_2/i_2$ satisfies

$$\bar{F} \circ i_1 = i_2 \quad \text{and} \quad p_2 \circ \bar{F} = p_1.$$

We will determine this equivalence relation of quadratic extensions with the help of a certain cohomology class, which we will introduce in section 4.

There is a natural notion of the direct sum of quadratic extensions. That is, if $(\mathfrak{g}_j, i_j, i_j, p_j)$, $j = 1, 2$ are quadratic extensions of \mathfrak{l}_j by α_j , then $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, i_1 \oplus i_2, i_1 \oplus i_2, p_1 \oplus p_2)$ is a quadratic extension of $\mathfrak{l}_1 \oplus \mathfrak{l}_2$ by $\alpha_1 \oplus \alpha_2$.

We call a direct sum non-trivial, if both summands are different from the trivial quadratic extension $(\{0\}, \{0\}, 0, 0)$ of $\{0\}$ by $\{0\}$. We call a quadratic extension decomposable, if it can be written as a non-trivial direct sum of two quadratic extensions. A quadratic extension, which is equivalent to a decomposable one, is also decomposable. Moreover, if a quadratic extension (\mathfrak{g}, i, i, p) is decomposable, then the corresponding metric symplectic Lie algebra \mathfrak{g} is decomposable as a metric symplectic Lie algebra. Conversely, we have the following lemma from [14] (see also [13]), since it does not depend on the semisimplicity of the derivation.

Lemma 14. ([14], see also [13]) *Let (\mathfrak{g}, i, i, p) be a balanced quadratic extension of \mathfrak{l} by α . The quadratic extension (\mathfrak{g}, i, i, p) is decomposable if and only if \mathfrak{g} is decomposable as a metric symplectic Lie algebra.*

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The task of this section is to introduce the cocycle, we will use to describe the quadratic extensions. Afterwards, we define the quadratic cohomology by using a certain group action on the set of cocycles.

Let $\rho : \mathfrak{l} \rightarrow \text{Hom}(\alpha)$ be a representation of the Lie algebra \mathfrak{l} on the vector space α . Let $C^p(\mathfrak{l}, \alpha) = \text{Hom}(\wedge^p \mathfrak{l}, \alpha)$ denote the space of alternating p -linear maps of \mathfrak{l} with values in α and $(C^*(\mathfrak{l}, \alpha), d)$ the standard Lie algebra cochain complex, where $d : C^p(\mathfrak{l}, \alpha) \rightarrow C^{p+1}(\mathfrak{l}, \alpha)$ is

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defined by

$$\begin{aligned} d\tau(L_1, \dots, L_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(L_i) \tau(L_1, \dots, \hat{L}_i, \dots, L_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \tau([L_i, L_j], L_1, \dots, \hat{L}_i, \dots, \hat{L}_j, \dots, L_{p+1}) \end{aligned}$$

for all $\tau \in C^p(\mathfrak{l}, \alpha)$. Moreover, let $Z^p(\mathfrak{l}, \alpha)$ and $B^p(\mathfrak{l}, \alpha)$ denote the groups of cocycles and coboundaries of $C^p(\mathfrak{l}, \alpha)$. The cochain complex of \mathfrak{l} with the trivial representation on \mathbb{R} is denoted by $C^*(\mathfrak{l})$.

Definition 15. Let $(\rho, \alpha, \langle \cdot, \cdot \rangle)$ be an orthogonal \mathfrak{l} -module. We define a bilinear multiplication $\langle \cdot \wedge \cdot \rangle : C^p(\mathfrak{l}, \alpha) \times C^q(\mathfrak{l}, \alpha) \rightarrow C^{p+q}(\mathfrak{l}, \mathbb{R})$ by

$$\langle \alpha \wedge \tau \rangle(L_1, \dots, L_{p+q}) = \sum_{[\sigma] \in \mathcal{S}_{p+q}/\mathcal{S}_p \times \mathcal{S}_q} \text{sgn}(\sigma) \langle \alpha(L_{\sigma(1)}, \dots, L_{\sigma(p)}), \tau(L_{\sigma(p+1)}, \dots, L_{\sigma(p+q)}) \rangle.$$

Here \mathcal{S}_k denotes the symmetric group of k letters.

Lemma 16. ([13, page 90]) Assume $\alpha \in C^p(\mathfrak{l}, \alpha)$ and $\tau \in C^q(\mathfrak{l}, \alpha)$. Then

$$d\langle \alpha \wedge \tau \rangle = \langle d\alpha \wedge \tau \rangle + (-1)^p \langle \alpha \wedge d\tau \rangle \text{ and} \quad (7)$$

$$\langle \alpha \wedge \tau \rangle = (-1)^{pq} \langle \tau \wedge \alpha \rangle. \quad (8)$$

We consider the pairs $(\mathfrak{l}, D_{\mathfrak{l}}, \alpha)$ of Lie algebras with derivations $(\mathfrak{l}, D_{\mathfrak{l}})$ and orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -modules (α, D_{α}) . This pairs form a category, whose morphisms are pairs (S, U) containing an homomorphism $S : \mathfrak{l}_1 \rightarrow \mathfrak{l}_2$ of Lie algebras and an isometric embedding $U : \alpha_2 \rightarrow \alpha_1$ satisfying $SD_{\mathfrak{l}_1} = D_{\mathfrak{l}_2}S$, $UD_{\alpha_2} = D_{\alpha_1}U$ and $U \circ \rho_2(SL) = \rho_1(L) \circ U$ for all $L \in \mathfrak{l}_1$.

We will denote the morphisms of this category by morphisms of pairs.

Remark 17. If (S, U) is a morphism from $(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \alpha_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \alpha_2)$, so it is a morphism from $(\mathfrak{l}_1, D_{\mathfrak{l}_{1s}}, \alpha_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_{2s}}, \alpha_2)$, where $D_{\mathfrak{l}_{is}}$ denotes the semisimple part of the Jordan decomposition of $D_{\mathfrak{l}_i}$ and $\alpha_i = (\alpha_i, D_{\alpha_{is}})$ the orthogonal $(\mathfrak{l}_i, D_{\mathfrak{l}_{is}})$ -module with the semisimple part $D_{\alpha_{is}}$ of the Jordan decomposition of D_{α_i} for $i = 1, 2$.

Remark 18. The category of pairs of Lie algebras and orthogonal modules was already introduced in [13]. The morphisms of that category were also called morphisms of pairs. In [14], morphisms of pairs were build, which also respect additional structure on the pairs of Lie algebras and orthogonal modules. A morphism of pairs from $(\mathfrak{l}_1, D_{\mathfrak{l}_{1s}}, \alpha_1)$ to $(\mathfrak{l}_2, D_{\mathfrak{l}_{2s}}, \alpha_2)$ in the sense of our definition, where $D_{\mathfrak{l}_{is}}$ is a semisimple derivation of \mathfrak{l}_i and $\alpha_i = (\alpha_i, D_{\alpha_{is}})$ an orthogonal $(\mathfrak{l}_i, D_{\mathfrak{l}_{is}})$ -module with semisimple $D_{\alpha_{is}}$ for $i = 1, 2$, is exactly the special case of the definition of morphisms of pairs of $(\mathbb{R}, \{e\})$ -equivariant metric Lie algebras in [14].

Definition 19. The direct sum of two pairs $(\mathfrak{l}_j, D_{\mathfrak{l}_j}, \alpha_j)$, $j = 1, 2$ is defined by

$$(\mathfrak{l}, D_{\mathfrak{l}}, \alpha) = (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \alpha_1) \oplus (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \alpha_2) := (\mathfrak{l}_1 \oplus \mathfrak{l}_2, D_{\mathfrak{l}_1} \oplus D_{\mathfrak{l}_2}, \alpha_1 \oplus \alpha_2),$$

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where \mathfrak{a}_1 and \mathfrak{a}_2 are orthogonal to each other, $D_{\mathfrak{a}_1} \oplus D_{\mathfrak{a}_2}$ is the derivation on $\mathfrak{a}_1 \oplus \mathfrak{a}_2$ and for $i \neq j$, $i, j = 1, 2$ the Lie algebra \mathfrak{l}_i acts trivially on \mathfrak{a}_j .

We call a direct sum non-trivial, if both summands are different from the trivial pair $(0, 0, 0)$.

Of course, if $(S, U) : (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ is an isomorphism of pairs and $(\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ a non-trivial direct sum of pairs, then $(\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1)$ is also a non-trivial direct sum of pairs.

Let $(S, U) : (\mathfrak{l}_1, D_{\mathfrak{l}_1}, \mathfrak{a}_1) \rightarrow (\mathfrak{l}_2, D_{\mathfrak{l}_2}, \mathfrak{a}_2)$ be a morphism of pairs. We define the following pull back maps

$$(S, U)^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) \rightarrow C^p(\mathfrak{l}_1, \mathfrak{a}_1), \quad (S, U)^* \alpha(L_1, \dots, L_p) := U \circ \alpha(S(L_1), \dots, S(L_p))$$

$$\text{and } (S, U)^* : C^p(\mathfrak{l}_2) \rightarrow C^p(\mathfrak{l}_1), \quad (S, U)^* \gamma(L_1, \dots, L_p) := \gamma(S(L_1), \dots, S(L_p)).$$

Lemma 20. ([13, page 92]) *The pull backs $(S, U)^*$ commute with the differential d and we have*

$$(S, U)^* \langle \alpha \wedge \tau \rangle = \langle (S, U)^* \alpha \wedge (S, U)^* \tau \rangle \quad (9)$$

for $\alpha \in C^p(\mathfrak{l}_2, \mathfrak{a}_2)$ and $\tau \in C^q(\mathfrak{l}_2, \mathfrak{a}_2)$.

Let \mathfrak{l} be a Lie algebra, $D_{\mathfrak{l}}$ a derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(e^{-tD_{\mathfrak{l}}}, e^{tD_{\mathfrak{a}}})$ is an isomorphism of pairs for every $t \in \mathbb{R}$. For $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$ denote $D^\circ \alpha = \frac{d}{dt}(e^{-tD_{\mathfrak{l}}}, e^{tD_{\mathfrak{a}}})^* \alpha|_{t=0}$. We obtain

$$D^\circ \alpha(L_1, \dots, L_p) = D_{\mathfrak{a}}(\alpha(L_1, \dots, L_p)) - \sum_{i=1}^p \alpha(L_1, \dots, D_{\mathfrak{l}} L_i, \dots, L_p).$$

For $\gamma \in C^p(\mathfrak{l})$ we get analogous

$$D^\circ \gamma(L_1, \dots, L_p) = - \sum_{i=1}^p \gamma(L_1, \dots, D_{\mathfrak{l}} L_i, \dots, L_p).$$

Lemma 21. *Suppose $\alpha \in C^p(\mathfrak{l}, \mathfrak{a})$, $\tau \in C^q(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^p(\mathfrak{l})$. Then we have*

$$D^\circ \langle \alpha \wedge \tau \rangle = \langle D^\circ \alpha \wedge \tau \rangle + \langle \alpha \wedge D^\circ \tau \rangle, \quad (10)$$

$$D^\circ d\gamma = dD^\circ \gamma, \quad (11)$$

$$D^\circ d\alpha = dD^\circ \alpha. \quad (12)$$

Proof. This follows from the properties of $(S, U)^*$. □

Remark 22. *The pull back maps and D° commute, i. e. $D_1^\circ(S, U)^* \gamma = (S, U)^* D_2^\circ \gamma$ and $D_1^\circ(S, U)^* \alpha = (S, U)^* D_2^\circ \alpha$.*

Now, we give a cohomology, which also respects the (not necessary semisimple) derivations.

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Let D_{ls} and D_{as} denote the semisimple parts of the Jordan decomposition of the derivations D_l of l and D_a of a for the rest of this work. The nilpotent parts are denoted by D_{ln} and D_{an} . Moreover, denote $D_s^\circ \alpha = \frac{d}{dt}(e^{-tD_{ls}}, e^{tD_{as}})^* \alpha \big|_{t=0}$.

Definition 23. Let p be even. We set (exactly as in [14])

$$Z_Q^p(l, \phi_l, a) := \left\{ (\alpha, \gamma) \in C^p(l, a) \oplus C^{2p-1}(l) \mid d\alpha = 0, d\gamma = \frac{1}{2} \langle \alpha \wedge \alpha \rangle, D_s^\circ \alpha = 0, D_s^\circ \gamma = 0 \right\}. \quad (13)$$

it is not hard to see that $(\alpha, \gamma) \in Z_Q^p(l, \phi_l, a)$ is invariant under the morphisms of pairs

$$(e^{tD_{ls}}, e^{-tD_{as}}), \quad t \in \mathbb{R}. \quad (14)$$

Let $C_Q^{p-1}(l, \phi_l, a) \subset C^{p-1}(l, a) \oplus C^{2p-2}(l)$ denote the set of tuples (τ, σ) , which are invariant under the morphisms of pairs (14). In our notation, this means that (τ, σ) satisfies

$$D_s^\circ \tau = 0 \quad \text{and} \quad D_s^\circ \sigma = 0. \quad (15)$$

Lemma 24. ([14, page 13], see also [13, Definition 1.1, Lemma 1.2]) The set $C_Q^{p-1}(l, \phi_l, a)$ becomes a group with group multiplication

$$(\tau_1, \sigma_1)(\tau_2, \sigma_2) := (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \frac{1}{2} \langle \tau_1 \wedge \tau_2 \rangle). \quad (16)$$

Moreover, suppose $(\alpha, \gamma) \in C^p(l, a) \oplus C^{2p-1}(l)$ and $(\tau, \sigma) \in C_Q^{p-1}(l, \phi_l, a)$. Then

$$(\alpha, \gamma)(\tau, \sigma) := \left(\alpha + d\tau, \gamma + d\sigma + \langle \alpha + \frac{1}{2} d\tau \wedge \tau \rangle \right) \quad (17)$$

defines a right action of the group $C_Q^{p-1}(l, \phi_l, a)$ on $C^p(l, a) \oplus C^{2p-1}(l)$, which leaves $Z_Q^p(l, \phi_l, a)$ invariant.

Now, we set the p -th. cohomology set $H_Q^p(l, \phi_l, a)$ of l with coefficients in a as

$$H_Q^p(l, \phi_l, a) := Z_Q^p(l, \phi_l, a) / C_Q^{p-1}(l, \phi_l, a).$$

For $(\alpha, \gamma) \in Z_Q^p(l, \phi_l, a)$ let $[\alpha, \gamma]$ denote the corresponding cohomology class.

Remark 25. The cohomology set $H_Q^p(l, \phi_l, a)$ was already studied in [14]. It was introduced to classify the equivalence classes of quadratic extensions of (\mathfrak{h}, K) -equivariant metric Lie algebras. Here this cohomology set is a special case and describes the equivalence classes of metric Lie algebras with semisimple skewsymmetric derivations. We shall detail this later.

Lemma 26. Let p be even and $(\delta, \epsilon), (\tau, \sigma) \in C_Q^{p-1}(l, \phi_l, a)$. Then

$$(\delta, \epsilon)(\tau, \sigma) := \left(\delta + D^\circ \tau, \epsilon + D^\circ \sigma + \langle \delta + \frac{1}{2} D^\circ \tau \wedge \tau \rangle \right) \quad (18)$$

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defines a right action of $C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ on $C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$.

Proof. Because of equation (10) we have

$$(\delta, \epsilon)((\tau_1, \sigma_1)(\tau_2, \sigma_2)) = ((\delta, \epsilon)(\tau_1, \sigma_1))(\tau_2, \sigma_2).$$

Since $D_s^\circ D^\circ = D^\circ D_s^\circ$, we obtain $D_s^\circ(\delta + D^\circ \tau) = 0$ and

$$D_s^\circ(\epsilon + D^\circ \sigma + \langle (\delta + \frac{1}{2} D^\circ \tau) \wedge \tau \rangle) = 0$$

for $(\delta, \epsilon), (\tau, \sigma) \in C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$. □

Definition 27. Let $Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$ denote the set of all $(\alpha, \gamma, \delta, \epsilon) \in Z_Q^p(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ satisfying $d\delta = D^\circ \alpha$ and $d\epsilon = D^\circ \gamma - \langle \alpha \wedge \delta \rangle$.

Lemma 28. Let p be even, $(\alpha, \gamma, \delta, \epsilon) \in Z_Q^p(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ and $(\tau, \sigma) \in C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$. Then

$$\begin{aligned} (\alpha, \gamma, \delta, \epsilon)(\tau, \sigma) &:= ((\alpha, \gamma)(\tau, \sigma), (\delta, \epsilon)(\tau, \sigma)) \\ &= (\alpha + d\tau, \gamma + d\sigma + \langle (\alpha + \frac{1}{2} d\tau) \wedge \tau \rangle, \delta + D^\circ \tau, \epsilon + D^\circ \sigma + \langle (\delta + \frac{1}{2} D^\circ \tau) \wedge \tau \rangle) \end{aligned} \quad (19)$$

defines a right action of $C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ on $Z_{Q+}^p(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a}) \oplus C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$, which leaves $Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$ invariant.

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$ and $(\tau, \sigma) \in C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$. Since the group action of $C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ leaves the cocycles $Z_Q^p(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a})$ invariant, it remains to show that the equations

$$\begin{aligned} d(\delta + D^\circ \tau) - D^\circ(\alpha + d\tau) &= 0 \text{ and} \\ \langle (\alpha + d\tau) \wedge (\delta + D^\circ \tau) \rangle + d(\epsilon + D^\circ \sigma + \langle \delta \wedge \tau \rangle + \frac{1}{2} \langle D^\circ \tau \wedge \tau \rangle) \\ &\quad - D^\circ(\gamma + d\sigma + \langle \alpha \wedge \tau \rangle + \frac{1}{2} \langle \tau \wedge d\tau \rangle) = 0 \end{aligned}$$

hold. The first equation follows directly from (12). Because of (7), (10) and the commutativity of d and D° the second equation is equivalent to

$$\langle \alpha \wedge \delta \rangle + d\epsilon - D^\circ \gamma + \langle (d\delta - D^\circ \alpha) \wedge \tau \rangle = 0.$$

Since $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$, this equation is satisfied. □

We set the p -th. quadratic cohomology $H_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$ of \mathbb{I} with coefficients in \mathfrak{a} as

$$H_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a}) := Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a}) / C_Q^{p-1}(\mathbb{I}, \phi_{\mathbb{I}}, \mathfrak{a}).$$

In addition, we denote the cohomology class of $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^p(\mathbb{I}, D_{\mathbb{I}}, \mathfrak{a})$ by $[\alpha, \gamma, \delta, \epsilon]$.

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Let $(l, D_l, a) = (l_1, D_{l_1}, a_1) \oplus (l_2, D_{l_2}, a_2)$ be the direct sum of two pairs. Let $j_i : a_i \rightarrow a$ and $q_i : l \rightarrow l_i$ denote the canonical embeddings and projections for $i = 1, 2$. The addition in $C^p(l, a) \oplus C^{2p-1}(l) \oplus C^{p-1}(l, a) \oplus C^{2p-2}(l)$ defines a map

$$+ : ((q_1, j_1)^* Z_{Q+}^p(l_1, D_{l_1}, a_1)) \times ((q_2, j_2)^* Z_{Q+}^p(l_2, D_{l_2}, a_2)) \rightarrow Z_{Q+}^p(l, D_l, a).$$

Since the addition respects the group action, we have a natural injective map

$$+ : ((q_1, j_1)^* H_{Q+}^p(l_1, D_{l_1}, a_1)) \times ((q_2, j_2)^* H_{Q+}^p(l_2, D_{l_2}, a_2)) \rightarrow H_{Q+}^p(l, D_l, a).$$

We call a cohomology class $[\alpha, \gamma, \delta, \epsilon] \in H_{Q+}^p(l, D_l, a)$ decomposable if there is a decomposition $(l, D_l, a) = (l_1, D_{l_1}, a_1) \oplus (l_2, D_{l_2}, a_2)$ into a non-trivial direct sum of pairs such that

$$[\alpha, \gamma, \delta, \epsilon] \in (q_1, j_1)^* H_{Q+}^p(l_1, D_{l_1}, a_1) + (q_2, j_2)^* H_{Q+}^p(l_2, D_{l_2}, a_2).$$

Otherwise, the cohomology class is called indecomposable. We denote the set of all indecomposable cohomology classes by $H_{Q+}^p(l, D_l, a)_i$.

Lemma 29. ([14, page 13], see also [13, Seite 92]) Let $(S, U) : (l_1, D_{l_{1s}}, a_1) \rightarrow (l_2, D_{l_{2s}}, a_2)$ be a morphism of pairs, where $D_{l_{is}}$ is semisimple and $(a_i, D_{a_{is}})$ orthogonal $(l_i, D_{l_{is}})$ -modules with semisimple derivations $D_{a_{is}}$ for $i = 1, 2$. Then

$$(S, U)^*(\alpha_2, \gamma_2) := ((S, U)^* \alpha_2, (S, U)^* \gamma_2)$$

defines a pull back map from $Z_Q^p(l_2, \phi_{l_2}, a_2)$ to $Z_Q^p(l_1, \phi_{l_1}, a_1)$.

Lemma 30. Let $(S, U) : (l_1, D_{l_1}, a_1) \rightarrow (l_2, D_{l_2}, a_2)$ be a morphism of pairs. Then

$$(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2) := ((S, U)^* \alpha_2, (S, U)^* \gamma_2, (S, U)^* \delta_2, (S, U)^* \epsilon_2) \quad (20)$$

defines a pull back map from $Z_{Q+}^p(l_2, D_{l_2}, a_2)$ to $Z_{Q+}^p(l_1, D_{l_1}, a_1)$.

Proof. From Lemma 29 we know that equation (20) defines a pull back from $Z_Q^p(l_2, \phi_{l_2}, a_2) \oplus C_Q^{p-1}(l_2, \phi_{l_2}, a_2)$ to $Z_Q^p(l_1, \phi_{l_1}, a_1) \oplus C_Q^{p-1}(l_1, \phi_{l_1}, a_1)$. Because of equation (9), (11), (12) and remark 22, we get

$$d(S, U)^* \delta_2 - D^\circ(S, U)^* \alpha_2 = 0$$

and

$$\langle (S, U)^* \alpha_2 \wedge (S, U)^* \delta_2 \rangle + d(S, U)^* \epsilon_2 - D^\circ(S, U)^* \gamma_2 = 0.$$

Thus $(S, U)^*$ defines a pull back map from $Z_{Q+}^p(l_2, D_{l_2}, a_2)$ to $Z_{Q+}^p(l_1, D_{l_1}, a_1)$. \square

Definition 31. Since the pull back map respects the group action in the sense

$$(S, U)^*((\alpha_2, \gamma_2, \delta_2, \epsilon_2)(\tau, \sigma)) = ((S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2))((S, U)^* \tau, (S, U)^* \sigma),$$

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we also have a pull back map from $H_{Q+}^p(l_2, D_{l_2}, \alpha_2)$ to $H_{Q+}^p(l_1, D_{l_1}, \alpha_1)$ given by

$$(S, U)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2] := [(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)].$$

Of course, the pull back map of isomorphisms of pairs also map decomposable cohomology classes to decomposable ones.

5 The standard model

In this section we define the canonical example $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ of a metric Lie algebra with skewsymmetric derivation for every $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(l, D_l, \alpha)$. Moreover, we show that this standard model is also a standard example of a quadratic extension and we describe this balanced quadratic extensions on the level of the quadratic cocycles. Therefor, we define the subset $Z_{Q+}^2(l, D_l, \alpha)_b$ of $Z_{Q+}^2(l, D_l, \alpha)$.

Example 32. Let $(l, [\cdot, \cdot]_l)$ be a Lie algebra, $(\rho, \alpha, \langle \cdot, \cdot \rangle_\alpha)$ an orthogonal l -module and $(\alpha, \gamma) \in C^2(l, \alpha) \oplus C^3(l)$. We define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $l^* \oplus \alpha \oplus l$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle = Z_1(L_2) + \langle A_1, A_2 \rangle_\alpha + Z_2(L_1)$$

for all $Z_1, Z_2 \in l^*$, $A_1, A_2 \in \alpha$ and $L_1, L_2 \in l$. Moreover, we consider a skewsymmetric bilinear map $[\cdot, \cdot] : l^* \oplus \alpha \oplus l \times l^* \oplus \alpha \oplus l \rightarrow l^* \oplus \alpha \oplus l$, which is given by

$$\begin{aligned} [l^*, l^* \oplus \alpha] &= 0, \\ [L_1, L_2] &= \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_l, \\ [L, A] &= -\langle A, \alpha(L, \cdot) \rangle + \rho(L)A, \\ [A_1, A_2] &= \langle \rho(\cdot)A_1, A_2 \rangle, \\ [L, Z] &= \text{ad}^*(L)(Z) = -Z([L, \cdot]_l) \end{aligned}$$

for all $Z \in l^*$, $A, A_1, A_2 \in \alpha$ and $L, L_1, L_2 \in l$.

This definition is exactly the definition of the standard model in [13].

Lemma 33. ([14, page 9]) Let (l, D_{l_s}) be a Lie algebra with semisimple derivation D_{l_s} and $(\rho, \alpha, \langle \cdot, \cdot \rangle, D_{\alpha_s})$ an orthogonal (l, D_{l_s}) -module, where D_{α_s} is semisimple.

Then $\mathfrak{d}_{\alpha, \gamma}(l, \alpha) := (l^* \oplus \alpha \oplus l, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$ is a metric Lie algebra with skewsymmetric derivation $D_{0,0}(D_{l_s}, D_{\alpha_s}) := -D_{l_s}^* \oplus D_{\alpha_s} \oplus D_{l_s}$ if and only if $(\alpha, \gamma) \in Z_Q^2(l, \phi_l, \alpha)$.

If l and α are clear from the context, we simply write $\mathfrak{d}_{\alpha, \gamma}$ for $\mathfrak{d}_{\alpha, \gamma}(l, \alpha)$.

Example 34. Let D_l and D_α be derivations of l and α . Consider $(\delta, \epsilon) \in C^1(l, \alpha) \oplus C^2(l)$, we define

$$D_{\delta, \epsilon}(D_l, D_\alpha) = \begin{pmatrix} -D_l^* & -\delta^* & \bar{\epsilon} \\ 0 & D_\alpha & \delta \\ 0 & 0 & D_l \end{pmatrix}.$$

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Here $\bar{\epsilon}$ is the uniquely determined linear map from \mathfrak{l} to \mathfrak{l}^* given by $\epsilon(L_1, L_2) = \langle \bar{\epsilon}(L_1), L_2 \rangle = \langle \bar{\epsilon}(L_1) \rangle(L_2)$ for $L_1, L_2 \in \mathfrak{l}$ and $\delta^* : \mathfrak{a} \rightarrow \mathfrak{l}^*$ is the dual map of δ given by $\langle \delta^* A, L \rangle = \langle A, \delta L \rangle$ for $L \in \mathfrak{l}$ and $A \in \mathfrak{a}$. If $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$ are clear from the context, we simply write $D_{\delta, \epsilon}$.

Let $i : \mathfrak{a} \rightarrow \mathfrak{a} \oplus \mathfrak{l}$ denote the canonical embedding and $p : \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{l}$ the canonical projection. If $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is a metric Lie algebra with skewsymmetric derivation, then $((\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})), \mathfrak{l}^*, i, p)$ is a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$, where we identify $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})/\mathfrak{l}^*$ and $\mathfrak{a} \oplus \mathfrak{l}$.

Let $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ also denote the quadratic extension of the standard model.

Theorem 2. *Let $(\mathfrak{l}, D_{\mathfrak{l}})$ be a Lie algebra with derivation $D_{\mathfrak{l}}$ and $(\rho, \alpha, \langle \cdot, \cdot \rangle, D_{\mathfrak{a}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. Then $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is a metric Lie algebra with skewsymmetric derivation, whose semisimple part of the Jordan decomposition of $D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is equal to $D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s})$, if and only if $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.*

Proof. Because of Lemma 33 the standard model $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ is a metric Lie algebra with skewsymmetric derivation, if and only if (α, γ) is an element in $Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Thus it remains to show that $D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is a skewsymmetric derivation of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$ if and only if the two conditions

$$d\delta = D^\circ \alpha, \quad (21)$$

$$d\epsilon = D^\circ \gamma - \langle \alpha \wedge \delta \rangle \quad (22)$$

hold and that $D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s})$ is exactly the semisimple part of $D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$, if and only if $D_s^\circ \delta = 0$ and $D_s^\circ \epsilon = 0$ is satisfied.

So, let $D_{\delta, \epsilon}$ be a skewsymmetric derivation of $\mathfrak{d}_{\alpha, \gamma}$. It holds $D[L_1, L_2] = [DL_1, L_2] + [L_1, DL_2]$ for $L_1, L_2 \in \mathfrak{l}$ if and only if

$$\begin{aligned} D_{\mathfrak{l}}[L_1, L_2]_{\mathfrak{l}} &= [D_{\mathfrak{l}}L_1, L_2]_{\mathfrak{l}} + [L_1, D_{\mathfrak{l}}L_2]_{\mathfrak{l}}, \\ D_{\mathfrak{a}}\alpha(L_1, L_2) &= -\delta[L_1, L_2]_{\mathfrak{l}} + \alpha(D_{\mathfrak{l}}L_1, L_2) - \rho(L_2)(\delta L_1) + \alpha(L_1, D_{\mathfrak{l}}L_2) + \rho(L_1)(\delta L_2), \\ \bar{\epsilon}[L_1, L_2]_{\mathfrak{l}} &= D_{\mathfrak{l}}^*\gamma(L_1, L_2, \cdot) + \gamma(D_{\mathfrak{l}}L_1, L_2, \cdot) + \langle \delta L_1, \alpha(L_2, \cdot) \rangle - \text{ad}^*(L_2)(\bar{\epsilon}L_1) \\ &\quad \delta^*\alpha(L_1, L_2) + \gamma(L_1, D_{\mathfrak{l}}L_2, \cdot) - \langle \delta L_2, \alpha(L_1, \cdot) \rangle + \text{ad}^*(L_1)(\bar{\epsilon}L_2) \end{aligned}$$

is satisfied for all $L_1, L_2 \in \mathfrak{l}$. These equations are equivalent to (21) and (22). On the other side it is easy to prove for $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ that $D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$ is a skewsymmetric derivation of $\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a})$, because of condition (21) and (22). Finally, $D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s})$ is the semisimple part of $D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}})$, if and only if

$$D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s})D_{\delta, \epsilon}(D_{\mathfrak{l}n}, D_{\mathfrak{a}n}) = D_{\delta, \epsilon}(D_{\mathfrak{l}n}, D_{\mathfrak{a}n})D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s})$$

holds, which is equivalent to $D_s^\circ \delta = 0$ and $D_s^\circ \epsilon = 0$. □

Again, we remark that nilpotent metric symplectic Lie algebra with bijective skewsymmetric derivations are our objects of interest to understand the metric, symplectic Lie

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algebras. So, we already defined balanced quadratic extensions only for this kind of metric Lie algebras with derivations in section 3 and now we again limit our observations for the rest of this section.

Definition 35. Let \mathfrak{l} be a nilpotent Lie algebra with bijective derivation $D_{\mathfrak{l}}$ and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module with bijective $D_{\mathfrak{a}}$. Let m denote the smallest positive integer such that $\mathfrak{l}^{m+2} = 0$. We define the set of all balanced cocycles $Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$ as the set of all cocycles $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$, which satisfy the following conditions for every $k = 0, \dots, m$:

(A_k) If there is an $A_0 \in \mathfrak{a}$ and $Z_0 \in (\mathfrak{l}^{k+1})^*$ for a given $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^{k+1}$ such that

$$\begin{aligned} \alpha(L, L_0) &= 0, \\ \gamma(L, L_0, \cdot) &= -\langle A_0, \alpha(L, \cdot) \rangle_{\mathfrak{a}} + \langle Z_0, [L, \cdot]_{\mathfrak{l}} \rangle \text{ as an element of } (\mathfrak{l}^{k+1})^* \end{aligned}$$

is satisfied for all $L \in \mathfrak{l}$, then $L_0 = 0$.

(B_k) The subspace $\alpha(\ker[\cdot, \cdot]_{\mathfrak{l} \otimes \mathfrak{l}^{k+1}}) \subset \mathfrak{a}$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$.

Moreover, let $Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ denote the set of cocycles $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$, where $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$.

The set $Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$ is used to describe on the level of quadratic cocycles when standard models of nilpotent metric Lie algebras define balanced quadratic extensions. Here we have:

Lemma 36. ([14], see also [12]) Let $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ be a nilpotent, metric Lie algebra with bijective derivation. The quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ is balanced if and only if $(\alpha, \gamma) \in Z_Q^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})_b$.

The property that a quadratic extension is balanced is a property of the corresponding metric Lie algebra and does not depend on the derivation. Thus, we obtain the following lemma:

Lemma 37. Let $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ be a nilpotent, metric Lie algebra with bijective derivation and let $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ be given. The quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ of a nilpotent Lie algebra \mathfrak{l} with bijective derivation $D_{\mathfrak{l}}$ by a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module $(\mathfrak{a}, D_{\mathfrak{a}})$ with bijective $D_{\mathfrak{a}}$ is balanced if and only if $(\alpha, \gamma, \delta, \epsilon)$ is an element in $Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$.

6 Equivalence to the standard model

The notation standard model for $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ comes from the fact that every quadratic extension of metric Lie algebras with skewsymmetric derivations is equivalent to $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ for a suitable $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. This, we will prove in this section.

Let $(\mathfrak{g}, D_s; i, i, p)$ be a quadratic extension of the metric Lie algebra with skewsymmetric semisimple derivation $(\mathfrak{l}, D_{\mathfrak{l}s})$ by an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}s})$ -module $(\mathfrak{a}, D_{\mathfrak{a}s})$, where $D_{\mathfrak{a}s}$ is

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semisimple. The subspace \mathfrak{i} is invariant under D_s , thus \mathfrak{i}^\perp is also invariant. Since D_s is semisimple and \mathfrak{i} isotropic, we can choose an isotropic complement of \mathfrak{i}^\perp , which is invariant under D_s . Let $s : \mathfrak{l} \rightarrow V_{\mathfrak{l}}$ be a section with $\tilde{p} \circ s = \text{id}$. Here $\tilde{p} : \mathfrak{g} \rightarrow \mathfrak{l}$ is the composition of the natural projection π from \mathfrak{g} to $\mathfrak{g}/\mathfrak{i}$ and p . This section satisfies $D_s \circ s = s \circ D_{\mathfrak{l}s}$ because of

$$\tilde{p} \circ D_s \circ s = p \circ \pi \circ D_s \circ s = p \circ \overline{D_s} \circ \pi \circ s = D_{\mathfrak{l}s} \circ \tilde{p} \circ s = \tilde{p} \circ s \circ D_{\mathfrak{l}s}.$$

Here we used that \tilde{p} is a bijection between \mathfrak{l} and $V_{\mathfrak{l}}$ and that p is a homomorphism from $(\mathfrak{g}/\mathfrak{i}, \overline{D_s})$ to $(\mathfrak{l}, D_{\mathfrak{l}s})$. Let $V_{\mathfrak{a}}$ be an orthogonal complement of $\mathfrak{i} \oplus s(\mathfrak{l})$ in \mathfrak{g} . We define $t : \mathfrak{a} \rightarrow V_{\mathfrak{a}}$ by

$$i(A) = t(A) + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}.$$

This map t is an isometry, since $i : \mathfrak{a} \rightarrow \mathfrak{i}^\perp/\mathfrak{i}$ is one. Moreover, we define an isomorphism $p^* : \mathfrak{l}^* \rightarrow \mathfrak{i}$ by

$$\langle p^*(Z), s(L) \rangle := \langle Z, (\tilde{p} \circ s)(L) \rangle = Z(L)$$

for all $Z \in \mathfrak{l}^*$ and $L \in \mathfrak{l}$. Then we define $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ and $\gamma \in C^3(\mathfrak{l})$ by

$$i(\alpha(L_1, L_2)) := [sL_1, sL_2] - s[L_1, L_2] + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}, \quad (23)$$

$$\gamma(L_1, L_2, L_3) := \langle [s(L_1), s(L_2)], s(L_3) \rangle. \quad (24)$$

Proposition 38. ([14], see also [13, Lemma 2.8 and proof]) It holds $(\alpha, \gamma) \in Z_{\mathbb{Q}}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$. Moreover, the map $\varphi := p^* + t + s : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{g}$ is an equivalence of the quadratic extensions $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}))$ and $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$.

Proposition 39. Let $(\mathfrak{g}, D; \mathfrak{i}, i, p)$ be a quadratic extension of $(\mathfrak{l}, D_{\mathfrak{l}})$ by $(\mathfrak{a}, D_{\mathfrak{a}})$. Then there is an $(\alpha, \gamma, \delta, \epsilon) \in Z_{\mathbb{Q}+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ such that the quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon})$ is equivalent to $(\mathfrak{g}, D; \mathfrak{i}, i, p)$. In addition $(\alpha, \gamma, \delta, \epsilon)$ is an element in $Z_{\mathbb{Q}+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ for an balanced quadratic extension.

Proof. We choose $(\alpha, \gamma) \in Z_{\mathbb{Q}}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ by equations (23) and (24) with respect to the quadratic extension $(\mathfrak{g}, D_s; \mathfrak{i}, i, p)$ of $(\mathfrak{l}, D_{\mathfrak{l}s})$ by $(\mathfrak{a}, D_{\mathfrak{a}s})$. Here D_s denotes the semisimple part of the Jordan decomposition of the derivation D (analogous $D_{\mathfrak{l}s}$ and $D_{\mathfrak{a}s}$ for $D_{\mathfrak{l}}$ and $D_{\mathfrak{a}}$). Proposition 38 says that $\varphi := p^* + t + s : \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l} \rightarrow \mathfrak{g}$ is an equivalence of quadratic extensions with semisimple parts of the derivations of $(\mathfrak{d}_{\alpha, \gamma}, D_{0,0}(D_{\mathfrak{l}s}, D_{\mathfrak{a}s}), \mathfrak{l}^*, i, p)$ to $(\mathfrak{g}, D_s, \mathfrak{i}, i, p)$. Now, we set

$$\langle A, \delta(L) \rangle_{\mathfrak{a}} := \langle t(A), D(s(L)) \rangle, \quad (25)$$

$$\epsilon(L_1, L_2) := \langle D(s(L_1)), s(L_2) \rangle. \quad (26)$$

Since φ is an isometry and D and $D_{\delta, \epsilon}$ are skewsymmetric, we get that $\varphi \circ D_{\delta, \epsilon} = D \circ \varphi$

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holds if and only if

$$\langle (D \circ \varphi)(L), X \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(L), X \rangle, \quad (27)$$

$$\langle (D \circ \varphi)(A_1), t(A_2) \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(A_1), t(A_2) \rangle \quad (28)$$

is satisfied for all $L \in \mathfrak{l}$, $X \in \mathfrak{g}$ and $A_1, A_2 \in \mathfrak{a}$. Because of

$$(\tilde{p} \circ D \circ t)(A_1) = (p \circ \pi \circ D \circ t)(A_1) = (p \circ \overline{D} \circ i)(A_1) = (D_{\mathfrak{l}} \circ p \circ i)(A_1) = 0$$

the element $(D \circ t)(A_1)$ lies in $V_{\mathfrak{a}} \oplus \mathfrak{i} = \mathfrak{i}^\perp$ for all $A_1 \in \mathfrak{a}$ and we obtain

$$\begin{aligned} \langle (D \circ \varphi)(A_1), t(A_2) \rangle &= \langle (D \circ t)(A_1), t(A_2) \rangle = \langle (\pi \circ D \circ t)(A_1), i(A_2) \rangle \\ &= \langle (\overline{D} \circ i)(A_1), i(A_2) \rangle_{\mathfrak{g}/\mathfrak{i}} = \langle (i \circ D_{\mathfrak{a}})(A_1), i(A_2) \rangle_{\mathfrak{g}/\mathfrak{i}} \\ &= \langle (t \circ D_{\mathfrak{a}})(A_1), t(A_2) \rangle = \langle (\varphi \circ D_{\delta, \epsilon})(A_1), t(A_2) \rangle. \end{aligned}$$

Here we used that i is a homomorphism of Lie algebras with derivations from $(\mathfrak{a}, D_{\mathfrak{a}})$ to $(\mathfrak{g}/\mathfrak{i}, \overline{D})$. Using

$$(\tilde{p} \circ D \circ s)(L_1) = (p \circ \overline{D} \circ \pi \circ s)(L_1) = (D_{\mathfrak{l}} \circ \tilde{p} \circ s)(L_1) = D_{\mathfrak{l}}(L_1)$$

for all $L_1 \in \mathfrak{l}$, we get furthermore

$$\begin{aligned} \langle (\varphi \circ D_{\delta, \epsilon})(L_1), p^*(Z_2) + t(A_2) + s(L_2) \rangle \\ &= \langle p^*\overline{\epsilon}(L_1), s(L_2) \rangle + \langle (t \circ \delta)(L_1), t(A_2) \rangle + \langle (s \circ D_{\mathfrak{l}})(L_1), p^*(Z_2) \rangle \\ &= \epsilon(L_1, L_2) + \langle \delta(L_1), A_2 \rangle_{\mathfrak{a}} + Z_2(D_{\mathfrak{l}}(L_1)) \\ &= \langle (D \circ s)(L_1), p^*(Z_2) + t(A_2) + s(L_2) \rangle \end{aligned}$$

for all $L_2 \in \mathfrak{l}$, $A_2 \in \mathfrak{a}$ and $Z_2 \in \mathfrak{l}^*$.

Finally, φ is an isomorphism from $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ to (\mathfrak{g}, D) , which maps $D_{0,0}(D_{\mathfrak{l}\mathfrak{s}}, D_{\mathfrak{a}\mathfrak{s}})$ to the semisimple part of D . Thus $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is a skewsymmetric derivation, whose semisimple part equals $D_{0,0}(D_{\mathfrak{l}\mathfrak{s}}, D_{\mathfrak{a}\mathfrak{s}})$. So $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ and φ an equivalence of quadratic extensions, because of Theorem 2.

If $(\mathfrak{g}, \mathfrak{i}, i, p)$ is balanced, then $(\mathfrak{d}_{\alpha, \gamma}, D_{\delta, \epsilon})$ is balanced, because of $\varphi(\mathfrak{l}^*) = \mathfrak{i}$. Using Lemma 37, we obtain that the cocycle $(\alpha, \gamma, \delta, \epsilon)$ is balanced. \square

7 Equivalence classes of quadratic extensions

The task of this section is to prove the bijection between $H_{Q^+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ and the equivalence classes of quadratic extensions of $(\mathfrak{l}, D_{\mathfrak{l}})$ by \mathfrak{a} .

Lemma 40. ([14], compare with the prove of Lemma 2.9 in [13]) *Let $(\mathfrak{l}, D_{\mathfrak{l}\mathfrak{s}})$ be a Lie algebra with semisimple derivation and $(\mathfrak{a}, D_{\mathfrak{a}\mathfrak{s}})$ an orthogonal $(\mathfrak{l}, D_{\mathfrak{l}\mathfrak{s}})$ -module, where $D_{\mathfrak{l}\mathfrak{s}}$ is semisimple.*

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Assume $(\alpha_i, \gamma_i) \in Z_{Q+}^2(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ for $i = 1, 2$. An isomorphism $F : \mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}, \mathfrak{a}) \rightarrow \mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}, \mathfrak{a})$ of Lie algebras is an equivalence of the quadratic extensions $(\mathfrak{d}_{\alpha_i, \gamma_i}(\mathfrak{l}, \mathfrak{a}), D_{0,0}(D_{\mathfrak{l}_S}, D_{\mathfrak{a}_S}))$ if and only if

$$F = \begin{pmatrix} \text{id} & -\tau^* & \bar{\sigma} - \frac{1}{2}\tau^*\tau \\ & \text{id} & \tau \\ & & \text{id} \end{pmatrix}$$

for a $(\tau, \sigma) \in C^1(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$ satisfying $\sigma(\cdot, \cdot) = \langle \bar{\sigma}(\cdot), \cdot \rangle$ and $(\alpha_1, \gamma_1) = (\alpha_2, \gamma_2)(\tau, \sigma)$. Here $\tau^* : \mathfrak{a} \rightarrow \mathfrak{l}^*$ denotes the dual map of τ defined by $\langle \tau^*A, L \rangle = \langle A, \tau L \rangle$ for $A \in \mathfrak{a}$ and $L \in \mathfrak{l}$.

Lemma 41. Let us suppose that $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ for $i = 1, 2$. The quadratic extensions $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$ are equivalent if and only if $[\alpha_1, \gamma_1, \delta_1, \epsilon_1] = [\alpha_2, \gamma_2, \delta_2, \epsilon_2] \in H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

Proof. Suppose $(\alpha_1, \gamma_1, \delta_1, \epsilon_1), (\alpha_2, \gamma_2, \delta_2, \epsilon_2) \in Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$. Then $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$ is a quadratic extension and $D_{0,0}(D_{\mathfrak{l}_S}, D_{\mathfrak{a}_S})$ the semisimple part of the Jordan decomposition of the derivation $D_{\delta_i, \epsilon_i}(D_{\mathfrak{l}_i}, D_{\mathfrak{a}_i})$.

For a bijection F from $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ the linear map $F \circ D_{0,0}(D_{\mathfrak{l}_S}, D_{\mathfrak{a}_S}) \circ F^{-1}$ is the semisimple part of the Jordan decomposition of D_{δ_2, ϵ_2} . Thus the map F is an equivalence of $(\mathfrak{d}_{\alpha_i, \gamma_i}, D_{\delta_i, \epsilon_i})$, $i = 1, 2$, if and only if it is an equivalence of the corresponding quadratic extensions from $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{0,0}(D_{\mathfrak{l}_S}, D_{\mathfrak{a}_S}))$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{0,0}(D_{\mathfrak{l}_S}, D_{\mathfrak{a}_S}))$ and $F \circ D_{\delta_1, \epsilon_1} = D_{\delta_2, \epsilon_2} \circ F$ holds. From Lemma 40 we obtain that the first equation is satisfied if and only if

$$F = \begin{pmatrix} \text{id} & -\tau^* & \bar{\sigma} - \frac{1}{2}\tau^*\tau \\ & \text{id} & \tau \\ & & \text{id} \end{pmatrix}$$

for a $(\tau, \sigma) \in C^1(\mathfrak{l}, \phi_{\mathfrak{l}}, \mathfrak{a})$, where $\sigma(\cdot, \cdot) = \langle \bar{\sigma}(\cdot), \cdot \rangle$ and $(\alpha_1, \gamma_1) = (\alpha_2, \gamma_2)(\tau, \sigma)$. Moreover, the condition $D_{\delta_2, \epsilon_2} \circ F = F \circ D_{\delta_1, \epsilon_1}$ holds, if

$$\langle D_{\delta_2, \epsilon_2} \circ FL, A \rangle = \langle F \circ D_{\delta_1, \epsilon_1} L, A \rangle \quad (29)$$

$$\langle D_{\delta_2, \epsilon_2} \circ FL_1, L_2 \rangle = \langle F \circ D_{\delta_1, \epsilon_1} L_1, L_2 \rangle \quad (30)$$

is satisfied for all $L, L_1, L_2 \in \mathfrak{l}$ and $A \in \mathfrak{a}$. Here equation (29) is equivalent to

$$\delta_1 = \delta_2 + D_{\mathfrak{a}} \circ \tau - \tau \circ D_{\mathfrak{l}}. \quad (31)$$

Equation (30) is equivalent to

$$\begin{aligned} \epsilon_1(L_1, L_2) &= \epsilon_2(L_1, L_2) + D^\circ \sigma(L_1, L_2) + \langle \delta_1 L_1, \tau L_2 \rangle \\ &\quad - \langle \tau L_1, \delta_2 L_2 \rangle + \frac{1}{2} \langle \tau \circ D_{\mathfrak{l}} L_1, \tau L_2 \rangle + \frac{1}{2} \langle \tau L_1, \tau \circ D_{\mathfrak{l}} L_2 \rangle, \end{aligned}$$

which leads to

$$(\delta_1, \epsilon_1) = (\delta_2, \epsilon_2)(\tau, \sigma)$$

with the use of equation (31). \square

Remark 42. Using Lemma 37 and Lemma 41 we get that the group action of $C_Q^1(l, \phi_l, \alpha)$ on $Z_{Q+}^2(l, D_l, \alpha)$ leaves the set of balanced cocycles $Z_{Q+}^2(l, D_l, \alpha)_b$ invariant.

For a nilpotent Lie algebra l with bijective derivation D_l and a trivial (l, D_l) -module (α, D_α) , where D_α is bijective, we define

$$H_{Q+}^2(l, D_l, \alpha)_b := Z_{Q+}^2(l, D_l, \alpha)_b / C_Q^1(l, \phi_l, \alpha). \quad (32)$$

Theorem 3. The Equivalence classes of quadratic extensions of (l, D_l) by (α, D_α) are in one-to-one correspondence with $H_{Q+}^2(l, D_l, \alpha)$. Moreover, the equivalence classes of balanced quadratic extensions are in bijection with $H_{Q+}^2(l, D_l, \alpha)_b$.

Proof. Let $(g, D; i, i, p)$ be a quadratic extension of (l, D_l) by (α, D_α) . Consider $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q+}^2(l, D_l, \alpha)$, which are given by the equations (23), (24), (25) and (26) with respect to two sections $s_i : l \rightarrow g$, $i = 1, 2$. Proposition 39 says that $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ and $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ are equivalent, since both are equivalent to $(g, D; i, i, p)$. Thus $[\alpha_1, \gamma_1, \delta_1, \epsilon_1] = [\alpha_2, \gamma_2, \delta_2, \epsilon_2]$, because of Lemma 41. This shows that the cohomology group of $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(l, D_l, \alpha)$ defined by Proposition 39 does not depend on the choice of the section s . Using Theorem 2, Proposition 39 and Lemma 41 we obtain the assertion. \square

8 Isomorphism classes of metric, symplectic Lie algebras

Until now, we know that we can represente the isomorphism classes of metric, symplectic Lie algebras by a standard model. Now, we determine when two standard models are isomorphic as metric, symplectic Lie algebras and, finally, give a classification scheme for metric, symplectic Lie algebras.

Proposition 43. ([14], see also the proof of Lemma 4.1 in [13]) Let (l_i, D_{l_i}) be Lie algebras with semisimple derivations D_{l_i} and (α_i, D_{α_i}) trivial (l_i, D_{l_i}) -modules with semisimple D_{α_i} . Suppose $(\alpha_i, \gamma_i) \in Z_Q^2(l_i, \phi_{l_i}, \alpha_i)_b$ for $i = 1, 2$. Let

$$F : (\mathfrak{d}_{\alpha_1, \gamma_1}(l_1, \alpha_1), D_{\delta_1, \epsilon_1}) \rightarrow (\mathfrak{d}_{\alpha_2, \gamma_2}(l_2, \alpha_2), D_{\delta_2, \epsilon_2})$$

be an isomorphism and $\bar{F} : \alpha_1 \oplus l_1 \rightarrow \alpha_2 \oplus l_2$ the corresponding isomorphism on the quotient induced by F . We set

$$S(L) := (p \circ \bar{F})(L), \quad U(A) := \bar{F}^{-1}(A)$$

for $L \in l_1$, $A \in \alpha_2$. Then (S, U) is an isomorphism of pairs of (l_1, D_{l_1}, α_1) and (l_2, D_{l_2}, α_2) .

Theorem 4. Suppose $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q+}^2(l_i, D_{l_i}, \alpha_i)_b$ for $i = 1, 2$. The metric Lie algebras with bijective skewsymmetric derivations $(\mathfrak{d}_{\alpha_i, \gamma_i}(l_i, \alpha_i), D_{\delta_i, \epsilon_i})$ are isomorphic if and only if there is an

isomorphism of pairs $(S, U) : (l_1, D_{l_1}, \alpha_1) \rightarrow (l_2, D_{l_2}, \alpha_2)$ such that

$$(S, U)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2] = [\alpha_1, \gamma_1, \delta_1, \epsilon_1] \in H_{Q+}^2(l_1, D_{l_1}, \alpha_1)_b.$$

Proof. Let (S, U) be an isomorphism of pairs such that $(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)$ lies in the cohomology set of $(\alpha_1, \gamma_1, \delta_1, \epsilon_1)$. The map

$$\varphi = \text{diag}(S^{*-1}, U^{-1}, S) : l_1^* \oplus \alpha_1 \oplus l_1 \rightarrow l_2^* \oplus \alpha_2 \oplus l_2 \quad (33)$$

defines an isomorphism of the Lie algebra $\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}(l_1, \alpha_1)$ with corresponding derivation $D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2}(D_{l_1}, D_{\alpha_1})$ to the Lie algebra $\mathfrak{d}_{\alpha_2, \gamma_2}$ with derivation D_{δ_2, ϵ_2} .

Since $(S, U)^*(\alpha_2, \gamma_2, \delta_2, \epsilon_2)$ and $(\alpha_1, \gamma_1, \delta_1, \epsilon_1)$ lie in the same cohomology set, the corresponding quadratic extensions are equivalent. Especially $(\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}, D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2})$ is isomorphic to $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$. Thus $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1})$ and $(\mathfrak{d}_{\alpha_2, \gamma_2}, D_{\delta_2, \epsilon_2})$ are isomorphic.

Now, suppose $(\alpha_i, \gamma_i, \delta_i, \epsilon_i) \in Z_{Q+}^2(l_i, D_{l_i}, \alpha_i)_b$, $i = 1, 2$. Let

$$F : (\mathfrak{d}_{\alpha_1, \gamma_1}(l_1, \alpha_1), D_{\delta_1, \epsilon_1}) \rightarrow (\mathfrak{d}_{\alpha_2, \gamma_2}(l_2, \alpha_2), D_{\delta_2, \epsilon_2})$$

be an isomorphism of the metric Lie algebras. Since the cocycles are balanced, we have $F(l_1^*) = l_2^*$ and F induces an isomorphism $\bar{F} : \alpha_1 \oplus l_1 \rightarrow \alpha_2 \oplus l_2$. Now, we set

$$S(L) := (p \circ \bar{F})(L), \quad U(A) := \bar{F}^{-1}(A)$$

for $L \in l_1$, $A \in \alpha_2$. Because of $D_{\delta_1, \epsilon_1} \circ F = F \circ D_{\delta_2, \epsilon_2}$ and Proposition 43 we obtain that (S, U) is an isomorphism of the pairs (l_i, D_{l_i}, α_i) . Thus, φ given by equation (33) defines an isomorphism from the metric Lie algebra with skewsymmetric bijective derivation $(\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}(l_1, \alpha_1), D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2})$ to $(\mathfrak{d}_{\alpha_2, \gamma_2}(l_2, \alpha_2), D_{\delta_2, \epsilon_2})$. Thus the corresponding quadratic extensions $(\mathfrak{d}_{(S,U)^*\alpha_2, (S,U)^*\gamma_2}, D_{(S,U)^*\delta_2, (S,U)^*\epsilon_2}, l_1^*, i, p)$ and $(\mathfrak{d}_{\alpha_1, \gamma_1}, D_{\delta_1, \epsilon_1}, l_1^*, i, p)$ are equivalent. Using Lemma 41 yields the assertion. \square

Remark 44. For an isomorphism of pairs (S, U) the map φ defined by equation (33) is an isomorphism of metric Lie algebras with derivations and it holds $\varphi(l_1^*) = l_2^*$. Thus, using Lemma 37 we obtain that the isomorphisms of pairs define pull back maps, which map the balanced cocycles of $Z_{Q+}^2(l_2, D_{l_2}, \alpha_2)$ to balanced cocycles of $Z_{Q+}^2(l_1, D_{l_1}, \alpha_1)$.

Lemma 45. Assume $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(l, D_l, \alpha)_b$. The quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(l, \alpha), D_{\delta, \epsilon}(D_l, D_\alpha))$ is decomposable if and only if $[\alpha, \gamma, \delta, \epsilon]$ is decomposable.

Proof. In general, if a quadratic extension is decomposable, then the corresponding cohomology class $[\alpha, \gamma, \delta, \epsilon]$ given by equations (23), (24), (25) and (26) equals

$$(q_1, j_1)^*[\alpha^1, \gamma_1, \delta_1, \epsilon_1] + (q_2, j_2)^*[\alpha^2, \gamma_2, \delta_2, \epsilon_2],$$

where $[\alpha_i, \gamma_i, \delta_i, \epsilon_i]$ are the corresponding cohomology classes of the direct summands of

the quadratic extension. This immediately implies that $[\alpha, \gamma, \delta, \epsilon] \in H^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ is decomposable for an decomposable quadratic extension $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$. Conversely, if $[\alpha, \gamma, \delta, \epsilon] \in H^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$ is decomposable, then

$$[\alpha, \gamma, \delta, \epsilon] = (q_1, j_1)^*[\alpha_1, \gamma_1, \delta_1, \epsilon_1] + (q_2, j_2)^*[\alpha_2, \gamma_2, \delta_2, \epsilon_2]$$

for certain $[\alpha_i, \gamma_i, \delta_i, \epsilon_i] \in H_{Q+}^2(\mathfrak{l}_i, D_{\mathfrak{l}_i}, \mathfrak{a}_i)$. Then the observation in the beginning of the proof and Lemma 41 yield that $(\mathfrak{d}_{\alpha, \gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta, \epsilon}(D_{\mathfrak{l}}, D_{\mathfrak{a}}))$ is equivalent to the direct sum $(\mathfrak{d}_{\alpha_1, \gamma_1}(\mathfrak{l}_1, \mathfrak{a}_1), D_{\delta_1, \epsilon_1}(D_{\mathfrak{l}_1}, D_{\mathfrak{a}_1})) \oplus (\mathfrak{d}_{\alpha_2, \gamma_2}(\mathfrak{l}_2, \mathfrak{a}_2), D_{\delta_2, \epsilon_2}(D_{\mathfrak{l}_2}, D_{\mathfrak{a}_2}))$ and hence is decomposable. \square

Let $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ denote the set of all morphisms of pairs (S, U) constituting of automorphisms S of \mathfrak{l} and isometries U of \mathfrak{a} , which satisfy $SD_{\mathfrak{l}} = D_{\mathfrak{l}}S$ and $UD_{\mathfrak{a}} = D_{\mathfrak{a}}U$. We simply write G , if $(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ is clear from the context. Let $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$ denote the set of all balanced and indecomposable cohomology classes in $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$.

The final result of this section is the following classification scheme. It follows directly from Lemma 3, Theorem 1, Proposition 39 and Theorem 4.

Theorem 5. *Let \mathfrak{l} be a (nilpotent) Lie algebra, $D_{\mathfrak{l}}$ a bijective derivation of \mathfrak{l} and $(\mathfrak{a}, D_{\mathfrak{a}})$ a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module with skewsymmetric bijective derivation $D_{\mathfrak{a}}$ of \mathfrak{a} . The set of isomorphism classes of indecomposable metric symplectic Lie algebras \mathfrak{g} with the following properties*

- $\mathfrak{i}(\mathfrak{g})^\perp / \mathfrak{i}(\mathfrak{g})$ together with the induced bijective skewsymmetric derivation is isomorphic to $(\mathfrak{a}, D_{\mathfrak{a}})$ as a metric Lie algebra with derivation and
- $\mathfrak{g} / \mathfrak{i}(\mathfrak{g})^\perp$ together with the induced bijective derivation is isomorphic to $(\mathfrak{l}, D_{\mathfrak{l}})$ as a Lie algebra with derivation

is in one-to-one correspondence to the $G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})$ -orbits of $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$.

The set of isomorphism classes of indecomposable metric symplectic Lie algebras is in bijection to the set

$$\coprod_{(\mathfrak{l}, D_{\mathfrak{l}}) \in \mathcal{L}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}} H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 / G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}),$$

where \mathcal{L} is a system of representatives of the isomorphism classes of nilpotent Lie algebras with bijective derivations and $\mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}$ a system of representatives of the isomorphism classes of abelian metric Lie algebras with bijective skewsymmetric derivations (considered as trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -modules).

Remark 46. Here, the union was taken over a system of representatives \mathcal{L} of the isomorphism classes of all Lie algebras \mathfrak{l} with bijective derivations $D_{\mathfrak{l}}$ of \mathfrak{l} . But often $(\mathfrak{l}, D_{\mathfrak{l}})$ does not have a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module \mathfrak{a} such that $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0$ is not empty. We call a Lie algebra \mathfrak{l} with bijective $D_{\mathfrak{l}}$ admissible if there is such a trivial $(\mathfrak{l}, D_{\mathfrak{l}})$ -module. The set of isomorphism classes of indecomposable metric symplectic Lie algebras is in one-to-one correspondence to the set

$$\coprod_{(\mathfrak{l}, D_{\mathfrak{l}}) \in \mathcal{L}_{ad}} \coprod_{\mathfrak{a} \in \mathcal{A}_{\mathfrak{l}, D_{\mathfrak{l}}}} H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_0 / G(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a}),$$

where \mathcal{L}_{ad} is a system of representatives of the isomorphism classes of admissible Lie algebras with bijective derivations and \mathcal{A}_{l,D_1} a system of representatives of isomorphism classes of abelian metric Lie algebras with bijective skewsymmetric derivations (considered as trivial (l, D_1) -modules).

9 Metric symplectic Lie algebras in special cases

In this section we determine metric symplectic Lie algebras in special cases up to isomorphism with the help of the classification scheme in section 8.

We call a basis $\{a_1, \dots, a_p, a_{p+1}, \dots, a_{p+q}\}$ of a orthonormal, if $a_i \perp a_j$ for $i \neq j$ and $\langle a_i, a_i \rangle_a = -1$ for $i = 1, \dots, p$ and $\langle a_j, a_j \rangle_a = 1$ for $j = p+1, \dots, p+q$. In this case (p, q) is called signature and p the index of $\langle \cdot, \cdot \rangle_a$. Let $\langle \cdot, \cdot \rangle_{p,q}$ denote the non-degenerate symmetric bilinear form with signature (p, q) on \mathbb{R}^{p+q} , which satisfies that the standard basis of the \mathbb{R}^{p+q} is an orthonormal basis. Then we call $\mathbb{R}^{p,q} := (\mathbb{R}^{p+q}, \langle \cdot, \cdot \rangle_{p,q})$ the standard pseudo-euclidian space. Of course we will denote $\mathbb{R}^{0,n}$ as \mathbb{R}^n . Often, we choose a Witt basis for $\mathbb{R}^{1,1}$. This is a basis $\{a_1, a_2\}$ of \mathbb{R}^2 of isotropic vectors, which satisfy $\langle a_1, a_2 \rangle_{1,1} = 1$.

For $z = a + ib \in \mathbb{C}$ we set

$$D_z := \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Moreover, denote $D_{z_1, \dots, z_n} = \text{diag}(D_{z_1}, \dots, D_{z_n})$ for $z_1, \dots, z_n \in \mathbb{C}$ the block diagonal matrix with the matrices D_{z_1} till D_{z_n} on the diagonal. Denote

$$N_b = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$$

for $b \in \mathbb{R}$.

For a basis $\{\sigma^1, \dots, \sigma^n\}$ of \mathfrak{l}^* we write abbrevitory $\sigma^{ij} := \sigma^i \wedge \sigma^j$ and $\sigma^{ijk} := \sigma^i \wedge \sigma^j \wedge \sigma^k$.

Theorem 6. *The only non-abelian, metric Lie algebra of dimension smaller than eight having symplectic forms is $\mathfrak{d}_{0,\sigma^{123}}(\mathbb{R}^3, 0)$. Its symplectic forms are given (up to isomorphism) by exactly one of the following derivations:*

$$D_{0,0}(\text{diag}(a, b, c), 0), \quad D_{0,0}(\text{diag}(N_b, -2b), 0), \quad D_{0,0}(\text{diag}(D_{b+id}, -2b), 0)$$

mit $a \leq b \leq c$, $a, b, c \neq 0$, $d > 0$ und $a + b + c = 0$.

Theorem 7. *The set of $(\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a}), D_{\delta,\epsilon}(D_l, D_a))$, where*

$\mathfrak{l} = \mathbb{R}^3$, $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$, $D_a = D_{is}$ and

- $D_l = \text{diag}(D_{b+is}, -b)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,

$\mathfrak{l} = \mathbb{R}^3$, $\mathfrak{a} = \mathbb{R}^{1,1}$, $D_a = \text{diag}(s, -s)$ and

- $D_l = \text{diag}(s - e, e, -s - e)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,
- $D_l = \text{diag}(-s, 2s, -3s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, \sigma^1 \otimes a_2, 0)$,

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- $D_l = \text{diag}(3s, -2s, s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, \sigma^3 \otimes a_1, 0)$,
 $l = \mathbb{R}^3$, $\mathfrak{a} = \mathbb{R}^{1,1}$, $D_a = \text{diag}(\pm s, \mp s)$ and
- $D_l = \text{diag}(N_{\pm s/2}, \mp \frac{3}{2}s)$, $(\alpha, \gamma, \delta, \epsilon) = (\sigma^{12} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0)$,
- $D_l \in \left\{ \text{diag}(\pm s, e, \mp s - e), \text{diag}(N_{\pm s}, \mp 2s), \text{diag}(\pm s, N_{\mp s/2}), \text{diag}(\pm s, D_{\mp s + id}) \right\}$,
 $(\alpha, \gamma, \delta, \epsilon) = (0, \sigma^{123}, \sigma^1 \otimes a_1, 0)$

with $a \leq b \leq c$, $a, b, c \neq 0$, $a + b + c = 0$, $d, s > 0$, $e \notin \{0, s, -s\}$ forms a system of representatives of the isomorphism classes of indecomposable non-abelian metric Lie algebras of dimension 8 with bijective skewsymmetric derivations. Here $\{a_1, a_2\}$ is a Witt basis for $\mathfrak{a} = \mathbb{R}^{1,1}$ and an orthonormal basis for $\mathfrak{a} = \mathbb{R}^2$ or $\mathbb{R}^{2,0}$. Moreover, denote $\{\sigma^1, \sigma^2, \sigma^3\}$ the dual basis of the standard basis of \mathbb{R}^3 .

Theorem 8. *There are no non-abelian metric symplectic Lie-Algebras of index smaller than three.*

Theorem 9. *The only indecomposable non-abelian metric symplectic Lie algebras with index 3 (up to isomorphism) are given as the six-dimensional metric Lie algebra $\mathfrak{d}_{0, \sigma^{123}}(\mathbb{R}^3, 0)$ with derivations $D_{0,0}(D_l, 0)$, where $D_l \in \{\text{diag}(a, b, c), \text{diag}(N_b, -2b), \text{diag}(D_{b+is}, -2b)\}$, or the eight-dimensional metric Lie algebra $\mathfrak{d}_{\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0}(\mathbb{R}^3, \mathbb{R}^2)$ with derivations $D_{0,0}(\text{diag}(D_{b+is}, -b), D_{is})$ for $a \leq b \leq c$, $a, b, c \neq 0$, $a + b + c = 0$ and $s > 0$. Here $\{a_1, a_2\}$ is an orthonormal basis of $\mathfrak{a} = \mathbb{R}^2$ and $\{\sigma^1, \sigma^2, \sigma^3\}$ the dual basis of the standard basis of \mathbb{R}^3 .*

Proof of all theorems. The computations are so overseeable that we don't have to restrict the computations to indecomposable metric symplectic Lie algebras. We simply compute all non-abelian metric symplectic Lie algebras and can easily decide afterwards which are indecomposable.

For every 4-dimensional non-abelian nilpotent Lie algebra l there is an $m \in \mathbb{N}$ such that l^{m+1} is one-dimensional. Thus $Z_{Q+}^2(l, D_l, 0)_b$ is empty for every derivation D_l of l , since (A_m) is not satisfied. For $l = \mathbb{R}^4$ and $\mathfrak{a} = \{0\}$ we have for every $\gamma \in C^3(l)$ an isomorphism of pairs (S, U) such that $(S, U)^*(0, \gamma) = (0, 0)$ or $(S, U)^*(0, \gamma) = (0, \sigma^2 \wedge \sigma^3 \wedge \sigma^4)$. Since both cocycles $(0, 0)$ and $(0, \sigma^2 \wedge \sigma^3 \wedge \sigma^4)$ do not satisfy (A_0) , the set $H_{Q+}^2(\mathbb{R}^4, D_l, 0)_b$ is empty.

Lemma 47. *The set of balanced cocycles $Z_{Q+}^2(\mathbb{R}, D_l, \mathfrak{a})_b$ and $Z_{Q+}^2(\mathbb{R}^2, D_l, \mathfrak{a})_b$ does not contain any elements for every bijective derivation D_l of \mathbb{R}^2 and for every abelian metric Lie algebra \mathfrak{a} with bijective skewsymmetric derivation D_a .*

Proof. Since $H_Q^2(\mathbb{R}, \phi_l, \mathfrak{a})$ contains only the element $[0, 0]$ for every metric vector space \mathfrak{a} and therefor does not have any balanced elements, the set $Z_{Q+}^2(\mathbb{R}, D_l, \mathfrak{a})_b$ is also empty. Now, suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^2, D_l, \mathfrak{a})$ and let D_{l_s} be diagonalizable over \mathbb{R} at the moment. I. e. there is a diagonal basis $\{L_1, L_2\}$ of \mathbb{R}^2 for D_{l_s} such that $D_{l_s}L_1 = \lambda_1 L_1$ and $D_{l_s}L_2 = \lambda_2 L_2$. Because of $D_s^\circ \alpha = 0$ we have

$$\begin{aligned}
 0 &= \langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle \\
 &= \langle D_{a_s}(\alpha(L_1, L_2)) - \alpha(D_{l_s}L_1, L_2) - \alpha(L_1, D_{l_s}L_2), \alpha(L_1, L_2) \rangle \\
 &= \langle -\alpha(D_{l_s}L_1, L_2) - \alpha(L_1, D_{l_s}L_2), \alpha(L_1, L_2) \rangle \\
 &= -(\lambda_1 + \lambda_2) \langle \alpha(L_1, L_2), \alpha(L_1, L_2) \rangle.
 \end{aligned}$$

So, on the one hand $\alpha = 0$ (which is in addition the case for $\lambda_1 + \lambda_2 = 0$, since $D^\circ \alpha = 0$ and D_α is bijective) or on the other hand $\alpha(\mathbb{R}^2, \mathbb{R}^2)$ is an one-dimensional isotropic subspace of α .

Now, suppose D_{l_s} is not diagonalizable over \mathbb{R} . This means that there is a basis $\{L_1, L_2\}$ of l such that $D_{l_s}L_1 = aL_1 + bL_2$ and $D_{l_s}L_2 = -bL_1 + aL_2$ for $a, b \in \mathbb{R}$, $b \neq 0$. Then analogous

$$0 = \langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle = -2a \langle \alpha(L_1, L_2), \alpha(L_1, L_2) \rangle.$$

For $\alpha \neq 0$ we obtain that $2a \neq 0$ is an eigenvalue of D_{α_s} because of $D_s^\circ \alpha(L_1, L_2) = 0$. Then $\alpha(l, l)$ is an one-dimensional isotropic subspace. Moreover, the 3-Form γ on \mathbb{R}^2 is trivial. Thus, for every of these cases the condition (A_0) or (B_0) is not satisfied and hence $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q^+}^2(\mathbb{R}^2, D_l, \alpha)_b$. \square

Since the dimension of l is limited by the index, we obtain directly Theorem 8 from Lemma 47. In addition Lemma 47 shows that there is no symplectic metric Lie algebra of dimension less than 6, except for abelian ones.

Denote \mathfrak{h}_3 the three-dimensional Heisenberg algebra given by $[e_1, e_2] = e_3$.

Lemma 48. *The set of cocycles $Z_{Q^+}^2(\mathfrak{h}_3, D_l, \alpha)$ does not contain any balanced elements for every bijective derivation D_l on the Heisenberg algebra \mathfrak{h}_3 and every abelian metric Lie algebra α with skewsymmetric bijective derivation.*

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q^+}^2(\mathfrak{h}_3, D_l, \alpha)$. If $\alpha(\mathfrak{h}_3, \mathfrak{z}(\mathfrak{h}_3)) = 0$, then condition (A_1) is not satisfied. Thus $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q^+}^2(\mathfrak{h}_3, D_l, \alpha)_b$.

Now, suppose $\alpha(\mathfrak{h}_3, \mathfrak{z}(\mathfrak{h}_3)) \neq 0$ and assume that D_{l_s} is diagonalizable over \mathbb{R} . Then there is a diagonal basis $\{L_1, L_2, L_3\}$ for D_{l_s} satisfying $[L_1, L_2] = L_3$ and $\alpha(L_1, L_3) \neq 0$. This means that

$$D_{l_s}L_1 = \lambda_1 L_1, \quad D_{l_s}L_2 = \lambda_2 L_2, \quad D_{l_s}L_3 = (\lambda_1 + \lambda_2)L_3,$$

where $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \neq 0$. Because of

$$D_{\alpha_s} \alpha(L_1, L_3) = D_s^\circ \alpha(L_1, L_3) + \alpha(D_{l_s}L_1, L_3) + \alpha(L_1, D_{l_s}L_3) = (2\lambda_1 + \lambda_2) \alpha(L_1, L_3)$$

we obtain that $2\lambda_1 + \lambda_2 \neq 0$ is an eigenvalue of D_{α_s} . Thus $\alpha(L_1, L_3) \neq 0$ is isotropic as an eigenvector for $2\lambda_1 + \lambda_2$, since D_{α_s} is skewsymmetric. Moreover,

$$\begin{aligned} 0 &= \langle D_{\alpha_s} \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle + \langle \alpha(L_1, L_3), D_{\alpha_s} \alpha(L_2, L_3) \rangle \\ &= (2\lambda_1 + \lambda_2) \langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle + (\lambda_1 + 2\lambda_2) \langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle \\ &= 3(\lambda_1 + \lambda_2) \langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle. \end{aligned}$$

Since D_l is bijective, it holds that $\lambda_1 + \lambda_2 \neq 0$ and hence $\langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle = 0$. Now, assume that D_{l_s} is not diagonalizable over \mathbb{R} . Then there is a basis $\{L_1, L_2, L_3\}$ of l , which satisfies $D_{l_s}L_1 = aL_1 + bL_2$, $D_{l_s}L_2 = -bL_1 + aL_2$, $D_{l_s}L_3 = 2aL_3$ for $a, b \neq 0$ and $[L_1, L_2] = L_3$, since the center of \mathfrak{h}_3 is invariant under D_l .

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For $i, j = 1, 2$ we consider

$$0 = \langle D_s^\circ \alpha(L_i, L_3), \alpha(L_j, L_3) \rangle$$

and obtain analogous that

$$\langle \alpha(L_1, L_3), \alpha(L_1, L_3) \rangle = \langle \alpha(L_1, L_3), \alpha(L_2, L_3) \rangle = 0.$$

So, $\alpha(L_1, L_3)$ is orthogonal to every element in $\alpha(\mathfrak{l}, \mathfrak{z}(\mathfrak{l}))$ and (B_1) is not satisfied. Thus $(\alpha, \gamma, \delta, \epsilon) \notin Z_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, \mathfrak{a})_b$. \square

Since $H_{Q+}^2(\mathfrak{l}, D_{\mathfrak{l}}, 0)_b$ is empty for every 4-dimensional Lie algebra \mathfrak{l} and the Lie algebras \mathbb{R} , \mathbb{R}^2 and also \mathfrak{h}_3 are not admissible, we obtain Theorem 6 by determining $H_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, 0)_b/G$. For Theorem 7 it remains to determine $H_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b/G$ with $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{1,1}, \mathbb{R}^{2,0}\}$ and Theorem 9 follows by calculating the G -orbits of $H_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathbb{R}^{2n})_b$ for $n \in \mathbb{N}$.

For $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ we have

$$\begin{aligned} 0 &= -\langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_3, L_4) \rangle - \langle \alpha(L_1, L_2), (D_s^\circ \alpha)(L_3, L_4) \rangle \\ &= \langle \alpha(D_{\mathfrak{l}s} L_1, L_2) + \alpha(L_1, D_{\mathfrak{l}s} L_2), \alpha(L_3, L_4) \rangle + \langle \alpha(L_1, L_2), \alpha(D_{\mathfrak{l}s} L_3, L_4) + \alpha(L_3, D_{\mathfrak{l}s} L_4) \rangle \end{aligned} \quad (34)$$

for vectors $L_1, \dots, L_4 \in \mathfrak{l}$ because of $D_s^\circ \alpha = 0$ and the skewsymmetry of $D_{\mathfrak{as}}$. Moreover, we have

$$0 = -\langle (D_s^\circ \alpha)(L_1, L_2), \alpha(L_1, L_2) \rangle = \langle \alpha(D_{\mathfrak{l}s} L_1, L_2) + \alpha(L_1, D_{\mathfrak{l}s} L_2), \alpha(L_1, L_2) \rangle. \quad (35)$$

Lemma 49. For $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$ the conditions

- $\alpha = 0$,
- $\gamma \neq 0$,
- $\text{tr}(D_{\mathfrak{l}}) = 0$

are equivalent.

Proof. Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_{\mathfrak{l}}, \mathfrak{a})_b$. Let $D_{\mathfrak{l}}$ be a bijective zero-trace matrix. At first, let $D_{\mathfrak{l}}$ be diagonalizable over \mathbb{R} . Denote $\{L_1, L_2, L_3\}$ the basis of eigenvectors of $D_{\mathfrak{l}s}$ for the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3 \neq 0$. Consider

$$\begin{aligned} 0 &= \langle \alpha(D_{\mathfrak{l}s} L_i, L_j) + \alpha(L_i, D_{\mathfrak{l}s} L_j), \alpha(L_p, L_k) \rangle + \langle \alpha(L_i, L_j), \alpha(D_{\mathfrak{l}s} L_p, L_k) + \alpha(L_p, D_{\mathfrak{l}s} L_k) \rangle \\ &= (\lambda_i + \lambda_j + \lambda_p + \lambda_k) \langle \alpha(L_i, L_j), \alpha(L_p, L_k) \rangle \end{aligned}$$

(compare to equation (34)). Because of $\lambda_1 + \lambda_2 + \lambda_3 = 0$ we have $\langle \alpha(\mathfrak{l}, \mathfrak{l}), \alpha(\mathfrak{l}, \mathfrak{l}) \rangle = 0$. Now (B_0) implies that $\alpha(\mathfrak{l}, \mathfrak{l})$ is non-degenerate with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ and thus $\alpha = 0$. If the zero-trace matrix $D_{\mathfrak{l}s}$ is not diagonalizable over \mathbb{R} , then there is a basis $\{L_1, L_2, L_3\}$ of \mathfrak{l} such that $D_{\mathfrak{l}s} L_1 = aL_1 + bL_2$, $D_{\mathfrak{l}s} L_2 = -bL_1 + aL_2$ and $D_{\mathfrak{l}s} L_3 = -2aL_3$ for $a, b \neq 0$. As above, we can

consider the equation

$$0 = \langle \alpha(D_{\text{Is}}L_i, L_j) + \alpha(L_i, D_{\text{Is}}L_j), \alpha(L_p, L_k) \rangle + \langle \alpha(L_i, L_j), \alpha(D_{\text{Is}}L_p, L_k) + \alpha(L_p, D_{\text{Is}}L_k) \rangle$$

for $i, j, k, l \in \{1, 2, 3\}$ and obtain $\langle \alpha(l, l), \alpha(l, l) \rangle = 0$. Thus $\alpha = 0$ because of (B_0) .

Now, suppose $\alpha = 0$. Using (A_0) we obtain $\gamma \neq 0$. Finally from $\gamma \neq 0$ follows directly that D_l has zero trace by using

$$0 = D_s^\circ \gamma = -\text{tr}(D_l)\gamma.$$

□

Suppose $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b$. If D_{Is} is not diagonalizable over \mathbb{R} , then we consider the complexification of the derivations and differential forms. Denote $\{L_1, L_2, L_3\}$ a basis of eigenvectors of D_{Is} for the corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3 \neq 0$. We consider

$$0 = (D_s^\circ \epsilon)(L_i, L_j) = -\epsilon(D_{\text{Is}}L_i, L_j) - \epsilon(L_i, D_{\text{Is}}L_j) = -(\lambda_i + \lambda_j)\epsilon(L_i, L_j). \quad (36)$$

Of course, $\alpha = 0$ and $\delta = 0$ for $\mathfrak{a} = 0$. So D_l is a zero-trace matrix and $\gamma = k\sigma^{123} \neq 0$, where $\{\sigma^1, \sigma^2, \sigma^3\}$ is the dual of the standard basis of \mathbb{R}^3 . Since the sum of two eigenvalues of D_{Is} is not zero, we have $\epsilon = 0$. Thus $(\alpha, \gamma, \delta, \epsilon) = (0, k\sigma^{123}, 0, 0)$ with $k \neq 0$. Finally $(k^{-\frac{1}{3}} \text{id}, \text{id}) \in G(\mathbb{R}^3, D_l, 0)$ and we get

$$(k^{-\frac{1}{3}} \text{id}, \text{id})^*(0, k\sigma^{123}, 0, 0) = (0, \sigma^{123}, 0, 0). \quad (37)$$

Now, assume $\mathfrak{a} \neq \{0\}$. At first, let us assume that $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b$ for a bijective zero-trace-matrix D_l . Then, again, $\alpha = 0$, $\gamma = k\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \neq 0$ and $\epsilon = 0$ because of equation (36). Now, consider

$$0 = (D_s^\circ \delta)(L_i) = D_{\mathfrak{a}s}(\delta(L_i)) - \delta(D_{\text{Is}}L_i) = D_{\mathfrak{a}s}\delta(L_i) - \lambda_i\delta(L_i). \quad (38)$$

The eigenvalues of $D_{\mathfrak{a}}$ are purely imaginary for $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$. But the eigenvalues of the zero-trace matrix D_{Is} can't be purely imaginary, so we have $\delta = 0$. Because of $(k^{-\frac{1}{3}} \text{id}, \text{id}) \in G(\mathbb{R}^3, D_l, 0)$ and equation (37) the cohomology class

$$H_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b / G(\mathbb{R}^3, D_l, \mathfrak{a})$$

consists of only one element for bijective zero-trace matrices D_l and $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$, which is represented by $[0, \sigma^{123}, 0, 0]$. The same holds for an arbitrary euclidian vector space \mathfrak{a} of even dimension.

Now, assume $\mathfrak{a} = \mathbb{R}^{1,1}$. If there is no real number, which is an eigenvalue of $D_{\mathfrak{a}s} = D_{\mathfrak{a}}$ and D_{Is} at the same time, then $\delta = 0$ because of equation (38). If there is such a real number, then this number is unique, since both eigenvalues $\pm s \neq 0$ of $D_{\mathfrak{a}}$ can't be eigenvalues of the bijective zero-trace matrix D_{Is} at the same time. Thus $\delta(l)$ is either $\{0\}$ or it spans a one-

dimensional eigenspace of D_a . Assume $\delta \neq 0$. If there is no eigenvector \hat{v} of D_l such that $\delta(\hat{v}) \neq 0$, then there is a vector v in the generalized eigenspace $\pm s$ of D_l satisfying $\delta(v) \neq 0$ because of equation (38). Especially the generalized eigenspace for $\pm s$ is two-dimensional and the generalized eigenspace for $\mp 2s$ is one-dimensional. The vector v is no eigenvector of D_l . Thus there is an eigenvector \tilde{v} of D_l with $D_l v = \pm s + \tilde{v}$. We define $\tau \in C^1(\mathbb{R}^3, \phi_l, \mathbb{R}^{1,1})$ by $\tau(\tilde{v}) = \delta(v)$, $\tau(v) = 0$ and $\tau(w) = 0$ for an eigenvector w for the eigenvalue $\mp 2s$. Then $D_s^\circ \tau = 0$ because of $D_s^\circ \delta = 0$ and

$$D^\circ \tau(v) = D_s^\circ \tau(v) - \tau(\tilde{v}) = -\delta(v).$$

Hence $(0, \gamma, \delta, 0)(\tau, 0) = (0, \gamma, 0, 0)$ and thus $(0, \gamma, \delta, 0)$ and $(0, \gamma, 0, 0)$ are equivalent. Using the morphism of pairs $(S, U) = (k^{-\frac{1}{3}} \text{id}, \text{id})$ we obtain

$$(S, U)^*(0, \gamma, 0, 0) = (0, \sigma^{123}, 0, 0).$$

Now, let there be a real eigenvector D_l , which spans the one-dimensional subspace $\delta(l)$. This case can't be reduced to the previous case, since this property is invariant under the G -action. Moreover, it is invariant under equivalence, since

$$(\delta + D^\circ \tau)(v) = \delta(v) + D_a \tau(v) - \tau(D_l(v)) = \delta(v) + D_{as} \tau(v) - \tau(D_{ls}(v)) = (\delta + D_s^\circ \tau)(v) = \delta(v)$$

holds for an eigenvector v of D_l . We choose an eigenvector $v_1 \in l$ of D_l , which satisfies $\delta(v_1) \neq 0$. Then we extend v_1 to a real Jordan or orthonormal basis $\{v_1, v_2, v_3\}$ of D_l satisfying $\delta(v_2) = \delta(v_3) = 0$. We obtain $\gamma = k\sigma^{123} \neq 0$. Furthermore, we choose a vector a_2 in \mathfrak{a} such that $a_1 := \delta(v_1)$ and a_2 form a Witt basis of $\mathfrak{a} = \mathbb{R}^{1,1}$. We set

$$(S, U) = (k^{-\frac{1}{3}} \text{id}, \text{diag}(k^{\frac{1}{3}}, k^{-\frac{1}{3}})) \in G(\mathbb{R}^3, D_l, \mathbb{R}^{1,1})$$

and obtain

$$(S, U)^*(0, \gamma, \delta, 0) = (0, \sigma^{123}, \delta, 0).$$

So this shows that every $(0, \gamma, \delta, 0)$ with an one-dimensional subspace $\delta(l)$, which is spanned by an eigenvector of D_l , lives in the same G -orbit. Finally we have shown that for a bijective zero-trace matrix D_l

$$H_{Q+}^2(\mathbb{R}^3, D_l, \mathbb{R}^{1,1})_b / G(\mathbb{R}^3, D_l, \mathbb{R}^{1,1})$$

consists of only one element, if no eigenvalue of D_{ls} is one of D_a . If there is an eigenvalue of D_{ls} , which is one of D_a , then the cohomology class has two elements.

Assume $(\alpha, \gamma, \delta, \epsilon) \in Z_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b$ for bijective D_l with $\text{tr}(D_l) \neq 0$. Then $\gamma = 0$ and thus $\alpha \neq 0$.

At first, we assume $\mathfrak{a} = \mathbb{R}^2, \mathbb{R}^{2,0}$. Using $D_s^\circ \alpha = 0$ we see that D_l has the eigenvalues $a + is$, $a - is$ and $-a$ for an $a \neq 0$, where $\pm is$ are the eigenvalues of D_a with $s > 0$. Since D_l has no

purely imaginary eigenvalues and, furthermore, the sum of two eigenvalues is not equal to zero, we get $\delta = 0$ and $\epsilon = 0$ (also compare to equation (36) and (38)).

Now, we consider $H_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b/G$, where

$$D_l = \begin{pmatrix} a & -s \\ s & a \\ & & -a \end{pmatrix}, \quad D_a = \begin{pmatrix} & -s \\ s & \\ & & \end{pmatrix}$$

with $s > 0$ and $a \neq 0$. We have $\alpha(e_1, e_2) = 0$, since $D_s^\circ \alpha = 0$. It also holds that $\alpha(e_1, e_3) \neq 0$ and $\alpha(e_2, e_3) \neq 0$ are linearly independent because of (A_0) . Moreover, $\alpha(e_1, e_3)$ and $\alpha(e_2, e_3)$ are orthogonal to each other, since

$$0 = \langle \alpha(D_{ls}e_1, e_3) + \alpha(e_1, D_{ls}e_3), \alpha(e_1, e_3) \rangle = -s \langle \alpha(e_2, e_3), \alpha(e_1, e_3) \rangle,$$

and they have the same length because of

$$\begin{aligned} 0 &= \langle \alpha(D_{ls}e_1, e_3) + \alpha(e_1, D_{ls}e_3), \alpha(e_2, e_3) \rangle + 0 = \langle \alpha(e_1, e_3), \alpha(D_{ls}e_2, e_3) + \alpha(e_2, D_{ls}e_3) \rangle \\ &= s \langle \alpha(e_2, e_3), \alpha(e_2, e_3) \rangle - s \langle \alpha(e_1, e_3), \alpha(e_1, e_3) \rangle. \end{aligned}$$

We normalize the vectors $\{\alpha(e_1, e_3), \alpha(e_2, e_3)\}$ to an orthonormal basis $\{v_1, v_2\}$ of \mathfrak{a} . Then $\alpha = \mu(\sigma^{13} \otimes v_1 + \sigma^{23} \otimes v_2)$ with $\mu \neq 0$ because of $D_s^\circ \alpha = 0$ and using

$$(S, U) = (\text{diag}(1, 1, \mu^{-1}), \text{id}) \in G$$

yields

$$(S, U)^*(\alpha, 0, 0, 0) = (\sigma^{13} \otimes v_1 + \sigma^{23} \otimes v_2, 0, 0, 0).$$

Hence, we can represent $H_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b/G$ by $[\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0]$. Here $\{a_1, a_2\}$ denotes the standard basis of \mathfrak{a} . For an arbitrary euclidian vectorspace \mathfrak{a} of even dimension with orthonormal basis $\{a_1, \dots, a_{2n}\}$ we have analogous

$$H_{Q+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b/G = \{[\sigma^{13} \otimes a_1 + \sigma^{23} \otimes a_2, 0, 0, 0]\}.$$

Now, assume $\mathfrak{a} = \mathbb{R}^{1,1}$. Because of $D_s^\circ \alpha = 0$ and (B_0) we know that the sum of two eigenvalues of D_{ls} equals s and another sum equals $-s$, where $\pm s$ denote the eigenvalues of D_a with $s > 0$. This especially implies that the eigenvalues of D_{ls} are real numbers. Since the sum of two eigenvalues is not zero, we obtain $\epsilon \equiv 0$.

Let D_l be semisimple. Since the eigenvalues of D_l are real numbers, there is a basis of eigenvectors $\{L_1, L_2, L_3\}$. This basis is w. l. o. g. given such that, the sum of the eigenvalues of L_1 and L_2 equals s and the sum of the eigenvalues of L_2 and L_3 equals $-s$. Since $\text{tr } D_l \neq 0$, we have $\gamma = 0$ and $\alpha(l, l) \neq 0$ is non-degenerate. Thus $\alpha(L_1, L_2)$ is an eigenvector of D_a for the eigenvalue s and $\alpha(L_2, L_3)$ an eigenvector of D_a for $-s$. Moreover, $\text{tr } D_l \neq 0$ implies that the sum of the eigenvalues of L_1 and L_3 is not equal to $\pm s$ and hence $\alpha(L_1, L_3) = 0$. W. l. o.

References

g. we choose $\{L_1, L_2, L_3\}$ such that $\{\alpha(L_1, L_2), \alpha(L_2, L_3)\}$ is a Witt basis of \mathfrak{a} .

If no eigenvalue of D_l is an eigenvalue of $D_{\mathfrak{a}}$, then $\delta = 0$, because of (38). If there is such an eigenvalue, then we have on the one hand $\delta = k\sigma^1 \otimes a_2$ for $-s$ being an eigenvalue of D_l or on the other hand $\delta = k\sigma^3 \otimes a_1$ for s being an eigenvalue of D_l , since $D^\circ \delta = 0$. Here $k \in \mathbb{R}$. For $\delta \neq 0$ the cocycles $(\alpha, 0, 0, 0)$ and $(\alpha, 0, \delta, 0)$ are not equivalent, nor they lie in the same G -orbit. Moreover, $(S, U)^*(\alpha, 0, \delta, 0) = (\alpha, 0, k^{-1}\delta, 0)$ for $(S, U) = (\text{diag}(k^{-1}, k, k^{-1}), \text{id}) \in G$. For a semisimple D_l , whose trace doesn't vanish, the cohomology class $H_{Q^+}^2(l, D_l, \mathfrak{a})_b/G$ consists of

- no elements, if one eigenvalue of $D_{\mathfrak{a}}$ is not the sum of two eigenvalues of D_l ,
- exactly one element, if both eigenvalue of $D_{\mathfrak{a}}$ are the sum of two eigenvalues of D_l , but there is no eigenvalue of D_l which is also an eigenvalue of $D_{\mathfrak{a}}$ and
- exactly two elements, otherwise.

For a D_l , which is not semisimple, we have

$$0 = (d\delta - D^\circ \alpha)(L_2, L_3) = D_s^\circ \alpha(L_2, L_3) - \alpha(L_1, L_3) = -\alpha(L_1, L_3).$$

Here $\{L_1, L_2, L_3\}$ denotes a Jordan basis of D_l , where $D_l L_1 = \pm \frac{s}{2} L_1$, $D_l L_2 = \pm \frac{s}{2} L_2 + L_1$ and $D_l L_3 = \mp \frac{3}{2} s L_3$. It is easy to see that there is a Jordan basis such that $\alpha(L_1, L_2)$ is an eigenvector for $\pm s$, $\alpha(L_2, L_3)$ is one for $\mp s$ and $\{\alpha(L_1, L_2), \alpha(L_2, L_3)\}$ is a Witt basis of $\mathbb{R}^{1,1}$. Since $\pm s$ is no eigenvalue of D_l , we obtain $\delta = 0$ using equation (38). Thus, in this case $H_{Q^+}^2(\mathbb{R}^3, D_l, \mathbb{R}^{1,1})_b/G$ consists of exactly one element.

At this moment we determined the G -orbits of $H_{Q^+}^2(\mathbb{R}^3, D_l, \mathfrak{a})_b$ for every derivation D_l of $l = \mathbb{R}^3$ and every trivial (l, D_l) -moduls $(\mathfrak{a}, D_{\mathfrak{a}})$, where $\mathfrak{a} \in \{0, \mathbb{R}^2, \mathbb{R}^{1,1}, \mathbb{R}^{2,0}, \mathbb{R}^{2n}, \}$ and $D_{\mathfrak{a}}$ is bijective. Choosing representatives of the conjugation classes of the derivations of l and \mathfrak{a} yield the theorems. □

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